Symmetry Groups:

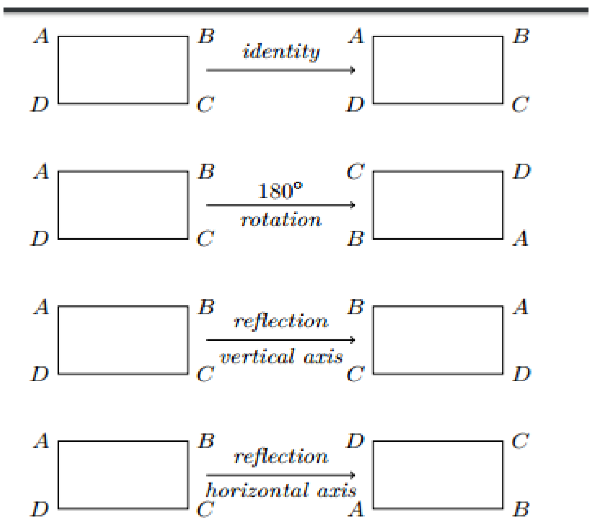


Figure 3.5: Rigid motions of a rectangle

- A $\underline{\text{symmetry}}$ of a geometric figure is a rearrangment of the figure which preserves the shape

- A map from the plane to itself preserving the symmentry of an object is called a $\underline{\text{rigid motion}}$.

sometimes just say symmetry as well.

Find all symmetries of an Equilateral triangle.

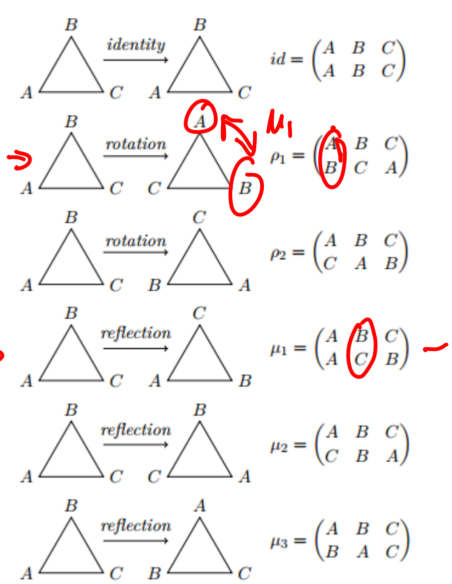
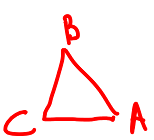


Figure 3.6: Symmetries of a triangle



$\underline{\text{Recall}}$: a $\underline{\text{permutation}}$ of a set S is a 1-1 and onto map $\pi: S \to S$

- 3 vertices $\Rightarrow$ $3! = 6$ permutations
  $\therefore$ Triangle has at most 6 symmetries

- Natural Question: what happens if we "multiply" or "compose" symmetries?

$\underline{\text{Ex}}$ what is $\mu_1 \rho_1 = \mu_2$ (Apply ρ_1 then apply μ_1)

$$\mu_1(\rho_1(A)) = \mu_1(B) = C$$

$$\mu_1 \rho_1 = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix} = \mu_2$$

Trying this with $\rho_1 \cdot \mu_1 = \mu_3 \neq \mu_1 \cdot \rho_1$

∘	id	ρ_1	ρ_2	μ_1	μ_2	μ_3
id	id	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	id	μ_3	μ_1	μ_2
ρ_2	ρ_2	id	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	id	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	id	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	id

This is a group

Table 3.7: Symmetries of an equilateral triangle

Formal Definition of a Group

A binary operation or law of composition on a set G is a function $G \times G \to G$ that assigns to each pair $(a,b) \in G \times G$ a unique element $a \cdot b$ or ab or $a \circ b$ or $a + b$ called the composition of a and b

A group $(G, \cdot)$ is a set G together with a binary operation $(a,b) \mapsto a \cdot b$ that satisfys the following axioms

- The binary operation is associative. That is
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
for any $a, b \in G$
- There exist an element $e \in G$ called the identity element such that $\forall\, a \in G$
$$e \cdot a = a \cdot e = a$$
- For each element $a \in G$ there exist an inverse element in G denoted a^{-1}, such that
$$a \cdot a^{-1} = a^{-1} \cdot a = e.$$

Note that $a \cdot b \neq b \cdot a$ in general

A group G s.t. $a \cdot b = b \cdot a \ \forall a, b \in G$ is called a abelian or commutative group.

Examples

- $(\mathbb{Z}_n, +)$ is a Group

  $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$

- $\mathbb{Z}_n$ is not a group with multiplication. Note $\mathbb{Z}_n \setminus \{0\}$ may not be a group either i.e. $2 \in \mathbb{Z}_6$

$$1 \cdot 2 = 2$$
$$2 \cdot 2 = 4$$
$$3 \cdot 2 = 0$$
$$4 \cdot 2 = 2$$
$$5 \cdot 2 = 4$$

things with a multiplicative inverse

Group of units of $\mathbb{Z}_n$

- $U(n) = \{ k \in \mathbb{Z}_n \mid \gcd(k, n) = 1 \}$

Ex| $U(8)$

$\cdot$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Table 3.12: Multiplication table for $U(8)$

A matrix Group

$$GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \det(A) \neq 0 ,\ a,b,c,d \in \mathbb{R} \right\}$$

$$\Updownarrow$$

$$\{ A \text{ 2x2 Real matrix} \mid A \text{ is invertible} \}$$

$GL_2(\mathbb{R})$ is a group under matrix multiplication.

To see why:

- Matrix mult. is associative
- $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{R})$ is the identity

  i.e. $IA = AI = A$ for all $A \in GL_2(\mathbb{R})$

- The product of two invertable matrices is invertable
  (since $\det(A \cdot B) = \det(A) \cdot \det(B)$
  or $(AB)^{-1} = B^{-1}A^{-1}$)
- Inverses exist $\forall\ A \in GL_2(\mathbb{R})$ that is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\therefore$ $GL_2(\mathbb{R})$ is a Group

- Note $GL_2(\mathbb{R})$ is <u>not</u> commutive

Example 3.15. Let

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

where $i^2 = -1$. Then the relations $I^2 = J^2 = K^2 = -1$, $IJ = K$, $JK = I$, $KI = J$, $JI = -K$, $KJ = -I$, and $IK = -J$ hold. The set $Q_8 = \{\pm 1, \pm I, \pm J, \pm K\}$ is a group called the ***quaternion group***. Notice that Q_8 is noncommutative.