Equivilence Relations and Partitions:

$\uparrow$ generalize the idea of equality

An <u>equivalence relation</u> on a set X is a relation $R \subseteq X \times X$ s.t.

- $(x,x) \in R \quad \forall x \in X$ (reflexive Property)
- $(x,y) \in R \Rightarrow (y,x) \in R$ (symetric property)
- $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$ (transitive property)

Often write $(x,y) \in R \Leftrightarrow x \sim y$

- $x \sim x$
- $x \sim y \Rightarrow y \sim x$
- $x \sim y$ and $y \sim z \Rightarrow x \sim z$

A <u>partition</u> P of a set X is a collection of non-empty sets $X_1, X_2, \ldots$ s.t. $X_i \cap X_j = \emptyset$ for $i \neq j$

and $\bigcup_k X_k = X$

<u>Equivilence class</u>

$[x] = \{ y \in X : y \sim x \}$

$\uparrow$ x is the repersentive of the eq. class.

Theorem 1.25. *Given an equivalence relation $\sim$ on a set X, the equivalence classes of X form a partition of X. Conversely, if $\mathcal{P} = \{X_i\}$ is a partition of a set X, then there is an equivalence relation on X with equivalence classes X_i.*

Proof:

First suppose $\exists$ eq. relation $\sim$ on X

$\Rightarrow$ for any $x \in X$ we have that $x \in [x]$ (by the reflexive property)

$\therefore$ $[x]$ is non-empty

$$X = \bigcup_{x \in X} [x]$$

Given $x, y \in X$ it remains to show that either

$$[x] = [y] \quad \text{or} \quad [x] \cap [y] = \emptyset$$

Suppose $[x] \cap [y]$ is non empty $\Rightarrow \exists z \in [x] \cap [y]$

$z \sim x$ and $z \sim y$ by symetry and transitivity

$$x \sim y \Rightarrow [x] \subseteq [y]$$

$$\text{and } y \sim x \Rightarrow [y] \subseteq [x]$$

$$[x] = [y]$$

$\therefore$ Two eq. classes are either disjoint or the same

Now suppose that $\mathcal{P} = \{X_i\}$ is a partition

Let $x \sim y$ if $x \in X_i$ and $y \in X_i$

$x \sim x$ ✓

If $x \sim y \Rightarrow y \sim x$ by def. If $\overbrace{x \in X_i \text{ and } y \in X_i}^{x \sim y}$

$\overbrace{y \in X_i \text{ and } z \in X_i}^{y \sim z} \Rightarrow x \sim z$ ✓

∎

Cor 1 Two eq. classes are either disjoint or equal

Ex 1 Let $p, q, r, s \in \mathbb{Z}$, $q \neq 0$, $s \neq 0$

Define

$$\frac{p}{q} \sim \frac{r}{s} \quad \text{if} \quad ps = qr \qquad \left(\text{i.e. } \frac{1}{2} \sim \frac{2}{4}\right)$$

$\frac{p}{q} \sim \frac{p}{q}$ ∴ reflexive $\qquad \frac{p}{q} \sim \frac{r}{s} \Rightarrow \frac{r}{s} \sim \frac{p}{q}$ ∴ symetric

Suppose $\frac{p}{q} \sim \frac{r}{s}$ and $\frac{r}{s} \sim \frac{t}{u}$ $\quad (q, s, u \neq 0)$

$$\Downarrow \qquad\qquad \Downarrow$$

Then $ps = qr$ and $ru = st$

$\cdot u \qquad q$

$$\Rightarrow \quad \underbrace{psu = qru = qst}$$

$$psu = qst \Rightarrow pu = qt \Rightarrow \frac{p}{q} \sim \frac{t}{u}.$$

(p, q) and (r, s) are in the same class if they reduce to the same fraction in lowest terms

Ex 2. Let r, s be in $\mathbb{Z}$, suppose $n \in \mathbb{N}$

r is <u>congruent</u> to s <u>modulo</u> n

or

$$r \equiv s \pmod n \qquad \left(r = s \pmod n \right)$$

If $r - s = n \cdot k$ for some $k \in \mathbb{Z}$

(alt. $r - s$ is evenly divisable by n)

i.e. $41 \equiv 17 \pmod 8$ since $41 - 17 = 24 = 8 \cdot 3$

Congruence mod n forms an Eq. relation on $\mathbb{Z}$

- $r \equiv r \pmod n$ since $r - r = 0 = 0 \cdot n$
- $r \equiv s \pmod n \Rightarrow r - s = n \cdot k \Rightarrow s - r = n \cdot (-k)$

  $s \equiv r \pmod n$
- $r \equiv s \pmod n$ and $s \equiv t \pmod n$

  $\Rightarrow r - s = kn$ and $s - t = l \cdot n$ $(k, l \in \mathbb{Z})$

  $r - s + s - t = kn + ln$

  $r - t = n(k + l) \Rightarrow r \equiv t \pmod n$ $\therefore$ transitive.

Consider $\mathbb{Z}/3\mathbb{Z} \cong$ integers modulo 3, we have

$[0] = \{ \ldots, -3, 0, 3, 6 \ldots \}$

$[1] = \{ \ldots, -2, 1, 4, \ldots \}$

$[2] = \{ \ldots, -1, 2, 5, 8, \ldots \}$

Integers + Mathematical Induction

Principle 2.1 (First Principle of Mathematical Induction). *Let $S(n)$ be a statement about integers for $n \in \mathbb{N}$ and suppose $S(n_0)$ is true for some integer n_0. If for all integers k with $k \geq n_0$, $S(k)$ implies that $S(k+1)$ is true, then $S(n)$ is true for all integers n greater than or equal to n_0.*

How do we show that

$$(*) \quad 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \qquad \text{for any } n?$$

$n = 1$

$$1 = \frac{1 \cdot (1+1)}{2} \qquad \checkmark \text{ True } n = 1$$

By Induction we may assume this is true for n and show that it holds for $n+1$

Show: $(*)$ holds for $n+1$

$$\underbrace{1 + 2 + \cdots + n}_{= \frac{n(n+1)}{2}} + (n+1) = \frac{n(n+1)}{2} + n + 1$$

$$= \frac{n^2 + 3n + 2}{2} = \frac{(n+1)((n+1)+1)}{2}$$

Principle 2.5 (Second Principle of Mathematical Induction). *Let $S(n)$ be a statement about integers for $n \in \mathbb{N}$ and suppose $S(n_0)$ is true for some integer n_0. If $S(n_0), S(n_0 + 1), \ldots, S(k)$ imply that $S(k+1)$ for $k \geq n_0$, then the statement $S(n)$ is true for all integers $n \geq n_0$.*

A non-empty subset S of $\mathbb{Z}$ is <u>well-ordered</u> if S contains a least element

($\mathbb{Z}$ not well-ordered since no least element

$\mathbb{N}$ is well ordered)

Principle 2.6 (Principle of Well-Ordering). *Every nonempty subset of the natural numbers is well-ordered.*

The Principle of Well-Ordering is equivalent to the Principle of Mathematical Induction.

Lemma 2.7. *The Principle of Mathematical Induction implies that 1 is the least positive natural number.*

Proof:

$S = \{ n \in \mathbb{N} \mid n \geq 1 \} \Rightarrow 1 \in S \quad 1 \geq 1$

assume $n \in S \quad n \geq 1$

$n+1 \geq 1 \Rightarrow n+1 \in S \quad \therefore$ by induction all Natral numbers are ≥ 1.

Theorem 2.8. *The Principle of Mathematical Induction implies the Principle of Well-Ordering. That is, every nonempty subset of $\mathbb{N}$ contains a least element.*