Sets and Equivilence Relations

Def 1 A set X is a well-defined collection of objects

For any object x we can decide if $x \in X$ or $x \notin X$

Ex 1

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z},\ q \neq 0 \right\}$$

Subsets $A \subseteq B$ $A \subset B$

$A \subsetneq B$ (proper subset)

Equality of sets

$A = B$ if $A \subseteq B$ and $B \subseteq A$

ϕ - Empty set

Set Operations

Union : $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$, $\bigcup_{i=1}^{n} A_i = A_1 \cup \cdots \cup A_n$

Intersection : $A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$ $\bigcap_{i=1}^{n} A_i = A_1 \cap \cdots \cap A_n$

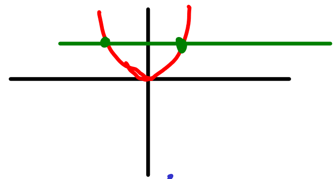
Disjoint if $A \cap B = \phi$

Suppose $A \subseteq U$

Complement : $A' = \{ x \mid x \in U \text{ and } x \notin A \}$

Difference : $A \setminus B = A \cap B' = \{ x \mid x \in A \text{ and } x \notin B \}$

Note $A' = U \setminus A$.

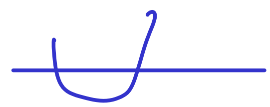


Ex) $U = \mathbb{R}^2$

$A = \{ (x,y) \mid y = x^2 \}$, $B = \{ (x,y) \mid y = 1 \}$

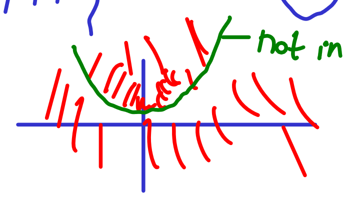
$A \cap B = \{ (1,1), (-1,1) \}$

$A \cup B = \{ (x,y) \mid y = x^2 \text{ or } y = 1 \} =$



$A \setminus B = \{ (x,y) \mid y = x^2 \text{ and } y \neq 1 \} =$

$A' = \{ (x,y) \mid y \neq x^2 \} =$



Proposition 1.2. *Let A, B, and C be sets. Then*

1. $A \cup A = A$, $A \cap A = A$, *and* $A \setminus A = \emptyset$;
2. $A \cup \emptyset = A$ *and* $A \cap \emptyset = \emptyset$;
3. $A \cup (B \cup C) = (A \cup B) \cup C$ *and* $A \cap (B \cap C) = (A \cap B) \cap C$;
4. $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$;
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Theorem 1.3 (De Morgan's Laws). *Let A and B be sets* and are subsets of a universal set U. *Then*

1. $(A \cup B)' = A' \cap B'$;
2. $(A \cap B)' = A' \cup B'$.

Proof: Show $(A \cup B)' \subseteq A' \cap B'$ and $A' \cap B' \subseteq (A \cup B)'$

First take $x \in (A \cup B)' \Rightarrow x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$ $\Rightarrow x \in A' \cap B'$

Now take $x \in A' \cap B'$ $\Rightarrow$ $x \in A'$ and $x \in B'$

$\Rightarrow x \notin A$ and $x \notin B$ $\Rightarrow$ $x \notin A \cup B$

$\Rightarrow x \in (A \cup B)'$ ∎

Cartesian product:

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

$$A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) : a_i \in A_i,\ i = 1, \ldots, n\}$$

Mappings

Subsets of $A \times B$ are called relations from a set A to a set B

A <u>mapping</u> or <u>function</u> from A to B is a relation

$f = \{(a,b) \mid \text{for } \forall a \in A \ \exists \text{ unique } b \in B\}$ i.e. f is a unique pairing

Note that not all $b \in B$ need appear in f

and

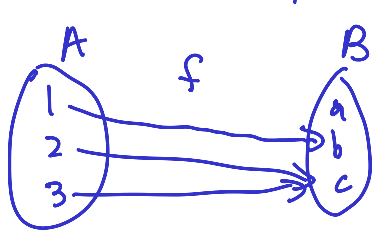
different $a_1, a_2 \in A$ can pair with the same b just NOT with multiple different b's.

$f : A \to B$ or $A \xrightarrow{f} B$

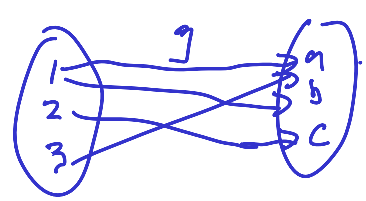
$f(a) = b$ or $f : a \mapsto b$ (instead $(a,b) \in f \subseteq A \times B$)

Range of f: $f(A) = \{f(a) \mid a \in A\} \subseteq B$

Ex 1



IS a function



Not a function

A Relation can fail to be a function if it is not well-defined

A Relation is well-defined if each element in the domain is assigned to a unique element in the Range.

Ex) $f: \mathbb{Q} \to \mathbb{Z}$ is NOT well-defined $\frac{1}{2} = \frac{2}{4}$

$: \frac{p}{q} \mapsto p$

$f(\frac{1}{2}) \neq f(\frac{2}{4}) = 2$

Onto / Surjective: $f(A) = B$, i.e. $\exists a \in A$ for each $b \in B$ s.t. $f(a) = b$

1-1 / Injective: $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

If $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

Bijective = 1-1 and onto.

Ex) $f: \mathbb{Z} \to \mathbb{Q}$

$: n \mapsto \frac{n}{1}$ is 1-1 but not onto.

$g: \mathbb{Q} \to \mathbb{Z}$

$: \frac{p}{q} \to p$ where $\frac{p}{q}$ is expressed in lowest terms and $q > 0$

is onto but NOT 1-1

Composition:

$f: A \to B$, $g: B \to C$

$g \circ f: A \to C$

$: x \mapsto g(f(x))$

Theorem 1.15. *Let $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then*

1. *The composition of mappings is associative; that is, $(h \circ g) \circ f = h \circ (g \circ f)$;*
2. *If f and g are both one-to-one, then the mapping $g \circ f$ is one-to-one;*
3. *If f and g are both onto, then the mapping $g \circ f$ is onto;*
4. *If f and g are bijective, then so is $g \circ f$.*

Identity mapping $\quad id_S : S \to S \quad$ (or id)
$\quad : s \mapsto s$

A map $g: B \to A$ is an inverse mapping of $f: A \to B$

if $\quad g \circ f = id_A \quad$ and $\quad f \circ g = id_B \quad g = f^{-1}$

Ex Suppose $S = \{1, 2, 3\}$ define a map

$$\pi(1) = 2, \quad \pi(2) = 3, \quad \pi(3) = 1$$

This is bijective. May also write this as

$$\begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (2\,3\,1)$$

For any set S a 1-1 and onto mappings $\pi: S \to S$ is called a permutation of S

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{or } (3\,1\,2)$$

$$\pi : 1 \mapsto 2, \quad \pi^{-1} : 2 \mapsto 1$$

$$\pi \circ \pi^{-1} = id_S$$

Theorem

A mapping is invertible iff it is both 1-1 and onto.