Def: A permutation is even if it can be expressed as an even number of transpositions.

The Alternating Group: (on n letters)

$\downarrow$

A_n = Set of all even permutations in S_n

Theorem 5.16 The set A_n is a subgroup of S_n.

Proof:

- product of two even permutation is even $\therefore$ A_n is closed
- id is even (Theorem from Friday) $\therefore$ $id \in A_n$
- If σ is even $\sigma = \sigma_1 \cdots \sigma_r$ for r even

$$(\sigma_1 \cdots \sigma_r)^{-1} = \sigma_r^{-1} \cdots \sigma_1^{-1} = \sigma_r \cdots \sigma_1$$

$$\therefore \sigma^{-1} \in A_n.$$

$\blacksquare$

Proposition 5.17

For $n \geq 2$ the number of even permutation is equal to the number of odd permutations $\Rightarrow$ $|A_n| = \frac{n!}{2}$

Proof:

A_n - even perm

B_n - odd perm

Show $\exists$ a bijection between A_n and B_n

Fix arbitrary transposition $\sigma \in S_n$ ($\exists$ since $n \geq 2$)

Define a map $\lambda_\sigma : A_n \to B_n$

$: \tau \mapsto \sigma \cdot \tau$

1-1 : Suppose $\lambda_\sigma(\tau) = \lambda_\sigma(\mu)$ for $\tau, \mu \in A_n$ then

$$\sigma\tau = \sigma\mu$$

$$\tau = \sigma^{-1}\sigma\tau = \sigma^{-1}\sigma\mu = \mu$$

$$\therefore \tau = \mu .$$

$$\therefore \lambda_\sigma \text{ is } 1\text{-}1$$

onto: Pick arbitrary $\alpha \in B_n$ Show $\exists$ $\tau \in A_n$ s.t $\lambda_\sigma(\tau) = \alpha$

Consider $\tau = \sigma\alpha$, Since α is odd $\therefore \tau$ is even and

$$\lambda_\sigma(\tau) = \sigma \cdot \sigma \cdot \alpha = \alpha$$

∎

Example 5.18. The group A_4 is the subgroup of S_4 consisting of even permutations. There are twelve elements in A_4:

(1)	(12)(34)	(13)(24)	(14)(23)
(123)	(132)	(124)	(142)
(134)	(143)	(234)	(243).

Dihedral Groups – Subgroups of S_n

n^{th} dihedral group : group of rigid motions of a regular n-gon

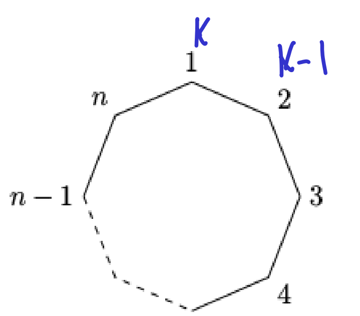


Figure 5.19: A regular n-gon

- Notice we have n choices for the first vertex
- If we replace 1 by k then 2 must be either k+1 or k−1
- 2n possible rigid motions (n reflections and n rotations)

Theorem 5.20

The dihedral group, D_n, is a subgroup of S_n of order $2n$

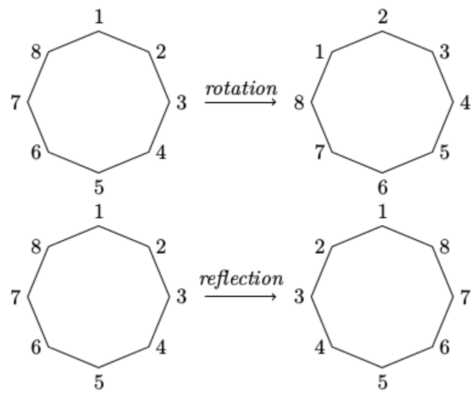


Figure 5.21: Rotations and reflections of a regular n-gon

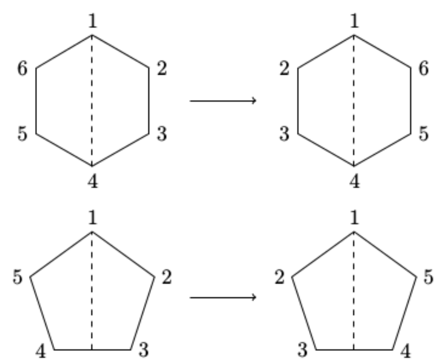


Figure 5.22: Types of reflections of a regular n-gon

Theorem 5.23

The group D_n, $n \geq 3$ consists of all products of two elements r and s satisfying the relations

$$r^n = 1 \quad \text{(rotations)}$$

$$s^2 = 1 \quad \text{(reflections)}$$

$$srs = r^{-1}$$

Proof: There are exactly n- rotations

$$id, \ \frac{2\pi}{n}, \ 2 \cdot \frac{2\pi}{n}, \ \ldots, \ (n-1)\frac{2\pi}{n}$$

$r = \frac{2\pi}{n}$ this generates all other rotations (think of roots of unity)

i.e. $r^k = k \cdot \frac{2\pi}{n}$

Label n reflections $s_1, \dots, s_n$ where s_k leaves the k^{th} vertex fixed. Two cases

Even # vertices

- Two vertices fixed by such a reflection

Odd # vertices

- one vertex fixed

$|s_k| = 2$

$s = s_1$ Then $s^2 = id$, $r^n = id$

Conside the first vertex of an n-gon:

Any rigid motion replace 1 by k

then 2 becomes either $k+1$ or $k-1$

if $2 \to k+1$ then

$$t = r^{k-1}$$

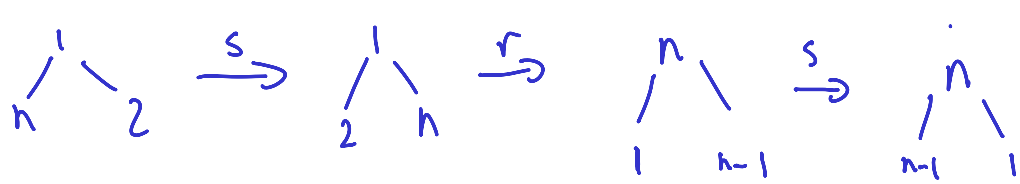
If 2 is replaced by $k-1$ then

$$t = r^{k-1} s$$

Show $r^{-1} = srs$

$r^{-1} = srs$

srs = First



Example | D_4 rigid motions of a square

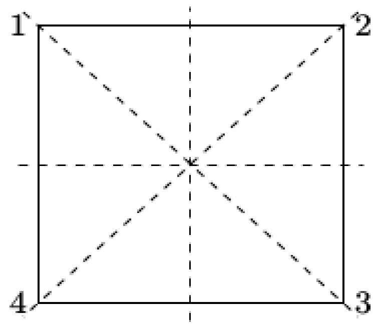


Figure 5.25: The group D_4

$|D_4| = 8$

rotations

$r = (1234)$

$r^2 = (13)(24)$

$r^3 = (1432)$

$r^4 = (1)$

Reflections

$s_1 = (24)$

$s_2 = (13)$

the other two reflections

$rs_1 = (12)(34)$

$r^3 s_1 = (14)(23)$ - reflection in "y" axis