Cycle Notation:

A permutation $\sigma \in S_X$ is a cycle of length k if $\exists$ $a_1, \ldots, a_k \in X$ s.t

$$\sigma(a_1) = a_2$$
$$\sigma(a_2) = a_3$$
$$\vdots$$
$$\sigma(a_k) = a_1$$

and $\sigma(x) = x$ for $x \neq a_i$ we write this

$$(a_1, a_2, \ldots, a_k)$$

Example 5.5. The permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 5 & 1 & 4 & 2 & 7 \end{pmatrix} = (162354)$$

is a cycle of length 6, whereas

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 3 & 5 & 6 \end{pmatrix} = (243)$$

is a cycle of length 3.

Not every permutation is a cycle. Consider the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} = (1243)(56).$$

This permutation actually contains a cycle of length 2 and a cycle of length 4.

Ex1 Products of cycles

$$\sigma = (1352), \quad \tau = (256)$$

σ: $1 \mapsto 3$, $3 \mapsto 5$, $5 \mapsto 2$, $2 \mapsto 1$

τ: $2 \mapsto 5$, $5 \mapsto 6$, $6 \mapsto 2$.

Compute $\sigma\tau$

- $\tau: 1 \mapsto 1$, $\sigma: 1 \mapsto 3$ $\Rightarrow$ $\sigma\tau: 1 \mapsto 3$
- $\tau: 2 \mapsto 5$, $\sigma: 5 \mapsto 2$ $\Rightarrow$ $\sigma\tau: 2 \mapsto 2$
- $\tau: 3 \mapsto 3$, $\sigma: 3 \mapsto 5$ $\Rightarrow$ $\sigma\tau: 3 \mapsto 5$

$\sigma\tau: 4 \mapsto 4$

$\sigma\tau: 5 \mapsto 6$, $\sigma\tau: 6 \mapsto 1$

$\sigma = (1352)$, $\tau = (256)$

$\sigma\tau = (1356)$

Two cycles $\sigma = (a_1, \ldots, a_k)$, $\tau = (b_1, \ldots, b_\ell)$ are <u>disjoint</u>

if $a_i \neq b_j \quad \forall i,j$

<u>Ex</u> (135) and (27) are disjoint

(135) and (347) are not disjoint

$(135)(27) = (135)(27)$

$(135)(347) = (13475)$

<u>Proposition 5.8</u>

Let σ and τ be two disjoint cycles in S_X. Then $\sigma\tau = \tau\sigma$

<u>Proof:</u>

Let $\sigma = (a_1, \ldots, a_k)$, $\tau = (b_1, \ldots, b_\ell)$

Show $\sigma\tau(x) = \tau\sigma(x) \quad \forall x \in X$.

If $x \notin \{a_1, \ldots, a_k\}$ and $x \notin \{b_1, \ldots, b_k\} \Rightarrow \sigma(x) = x$, $\tau(x) = x$

$\sigma\tau(x) = \sigma(\tau(x)) = x = \tau(\sigma(x)) = \tau\sigma(x)$

without loss of generality suppose $x \in \{a_1, \ldots, a_k\}$

So $x = a_i$ for some i $\quad \sigma(a_i) = a_{(i \bmod k)+1}$

$a_j \notin \{b_1, \ldots, b_\ell\} \therefore \quad \tau(a_j) = a_j$

$\sigma\tau(a_i) = \sigma(\tau(a_i)) = \sigma(a_i) = a_{(i \bmod k)+1} = \tau(a_{(i \bmod k)+1})$

$= \tau(\sigma(a_i))$

$= \tau\sigma(a_i)$

$\therefore \sigma\tau = \tau\sigma$ ∎

Theorem 5.9 Every permutation in S_n can be written as the product of disjoint cycles.

Proof

Assume $X = \{1, 2, \dots, n\}$. If $\sigma \in S_n$ set

$X_1 = \{\sigma(1), \sigma^2(1), \dots\}$: X_i is finite since X is

Let i be the first integer in X where $i \notin X_1$

$X_2 = \{\sigma(i), \sigma^2(i), \dots\}$

similarily we define $X_3, X_4, \dots$

Since X is finite we must eventually get all the elements in X. Say r steps

define cycles $\sigma_i(x) = \begin{cases} \sigma(x) & x \in X_i \\ x & x \notin X_i \end{cases}$

Then

$\sigma = \sigma_1 \cdot \sigma_2 \cdots \sigma_r$ since $X_1, \dots, X_r$ are disjoint. ∎

Example 5.10. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 1 & 5 & 2 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 5 & 6 & 4 \end{pmatrix}.$$

$\sigma = (1624)$, $\tau = (13)(456)$

Transpositions → A cycle of length 2.

Since

$(a_1, \dots, a_n) = (a_1 a_n)(a_1 a_{n-1}) \cdots (a_1 a_3)(a_1 a_2)$

Proposition 5.12. *Any permutation of a finite set containing at least two elements can be written as the product of transpositions.*

Note this is not unique, i.e.

$$(16)(253) = (16)(23)(25)$$
$$= (16)(45)(23)(45)(25)$$

However no permutation can be written as both an even and an odd number of transpositions.

<u>Lemma 5.14</u> If the identity is written as

$$id = \tau_1 \tau_2 \cdots \tau_r$$

for transposition $\tau_1, \ldots, \tau_r$ then <u>r is even.</u>

<u>Proof:</u> skip (use induction on r)

Theorem 5.15. *If a permutation σ can be expressed as the product of an even number of transpositions, then any other product of transpositions equaling σ must also contain an even number of transpositions. Similarly, if σ can be expressed as the product of an odd number of transpositions, then any other product of transpositions equaling σ must also contain an odd number of transpositions.*

<u>Proof:</u> (even case)

Suppose $\sigma = \sigma_1 \cdots \sigma_m = \tau_1 \cdots \tau_n$ (transpositions) where m is even

For a transposition ρ, $\rho^2 = id$

$$\sigma^{-1} = (\sigma_1 \cdots \sigma_m)^{-1} = \sigma_m^{-1} \cdots \sigma_1^{-1} = \sigma_m \cdots \sigma_1$$

$$id = \sigma\sigma^{-1} = \sigma\sigma_m \cdots \sigma_1 = \tau_1 \cdots \tau_n \cdot \sigma_m \cdots \sigma_1$$

$n+m$ is even, but m is even $\Rightarrow$ n is even. ∎