Homoromorphisms + Isomorphism Theorems

- N is a normal subgroup if $gH = Hg \ \forall g \in G$
- If N is a normal subgroup of G we can define

$$G/N = \{ aN \mid a \in G \}$$

Operation on G/N

$$(aN) \cdot (bN) = (abN)$$

G/N: factor group or quotient Group

aN: cosets of N

$$\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}_n$$

coset: $0 + n\mathbb{Z},\ 1 + n\mathbb{Z},\ 2 + n\mathbb{Z},\ \dots\ n-1 + n\mathbb{Z}$

Ex]

$$D_n / R_n \simeq \mathbb{Z}_2$$

$R_n = \langle r \rangle$

$$= \{ R_n, sR_n \}$$

$r^n = id$
$s^2 = id$
$rs = sr^{-1}$

- Homomorphism

$$\phi : G \longrightarrow H$$

$$\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2)$$

$\phi(e_G) = e_H$, $\phi(g^{-1}) = \phi(g)^{-1}$, $\phi(G)$ is a subgroup ...

$\ker(\phi) = $ all $g \in G$ s.t. $\phi(g) = e_H$.

$\uparrow$ Normal subgroup of G.

- Canonical homomorphisms (H a normal subgroup)

$$\phi : G \longrightarrow G/H$$
$$g \longmapsto gH$$

Note $\ker(\phi) = H$

<u>First Iso. Theorem</u>

G, H groups $\psi : G \to H$ a homomorphism,

$K = \ker\psi$ (is a normal subgroup of G). Let $\phi : G \to G/\ker\psi$ be the canonical homomorphism. $\exists$ unique isomorphism η

$\eta : G/\ker\psi \longrightarrow \psi(G)$ s.t. $\psi = \eta \circ \phi$

$$\begin{tikzcd}
G \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & H \\
& G/\ker\psi \arrow[ur, "\eta", "\sim"'] &
\end{tikzcd}$$

$\ker\psi = K$

$\Longleftrightarrow$ $\psi(G) = \eta \circ \phi(G)$

$\Updownarrow$

$\psi(G) \cong G/\ker(\psi)$

<u>Proof:</u> $K = \ker(\psi)$ is normal in G.

Define $\eta : G/K \longrightarrow \psi(G)$

$$gK \longmapsto \psi(g)$$

Show that η is well defined. If $g_1K = g_2K$ then we have

$k_1, k_2 \in K$

$$g_1 k_1 = g_2 k_2$$

$$g_1 \overbrace{k_1 k_2^{-1}}^{=K} = g_2$$

$$g_1 K = g_2$$

$\ker(\psi) = K$

identity

$$\eta(g_1K) = \psi(g_1) = \psi(g_1)\psi(K) = \psi(g_1K) = \psi(g_2) = \eta(g_2K)$$

$\therefore$ η is well defined.

Show η is a homomorphism

$$\eta(g_1K\, g_2K) = \eta(g_1g_2K) = \psi(g_1g_2)$$

$$= \psi(g_1)\psi(g_2)$$

$$= \eta(g_1K)\,\eta(g_2K)$$

η is onto by def.

Show 1-1. Say $\eta(g_1K) = \eta(g_2K)$

$$\Downarrow$$

$$\psi(g_1) = \psi(g_2)$$

$$e_H = (\psi(g_1))^{-1}\psi(g_2)$$

$$= \psi(g_1^{-1})\cdot\psi(g_2)$$

$$= \psi(g_1^{-1}g_2)$$

$$\Rightarrow e_G = g_1^{-1}g_2 \Rightarrow g_1 = g_2$$

$\therefore$ 1-1.

∎

Second isomorphism Theorem |

H subgroup of G, N normal subgroup of G

Then

- HN a subgroup of G
- $H \cap N$ - a normal subgroup of H
- $H/(H\cap N) \cong HN/N$.

Correspodence Theorem | H a subgroup, N a normal subgroup of G

$$H \longmapsto H/N$$

$$\{\text{Subgroups of } H \text{ containing } N\} \xleftrightarrow[\text{1-1}]{} \{\text{subgroups of } G/N\}$$

Third Iso theorem | G - a group, H, N - Normal subgroups

$$G/H \cong \frac{G/N}{H/N}.$$

Example | $\underset{\cong \mathbb{Z}_m}{\mathbb{Z}/m\mathbb{Z}} \cong (\mathbb{Z}/mn\mathbb{Z})/(m\mathbb{Z}/mn\mathbb{Z})$