Direct Products, continued

Corr Let $(g_1, \dots, g_n) \in \prod_{i=1}^{n} G_i$ and if $|g_i| = r_i < \infty$ in G_i

then $|(g_1, \dots, g_n)| = \operatorname{lcm}(r_1, \dots, r_n)$ in $\prod_{i=1}^{n} G_i$.

Ex $(8, 56) \in \mathbb{Z}_{12} \times \mathbb{Z}_{60}$

$|8| = 3$ $\qquad |56| = 15 = \dfrac{60}{\gcd(56, 60)}$

$|(8, 56)| = \operatorname{lcm}(3, 15) = 15.$

Ex $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \langle (1,1) \rangle \cong \mathbb{Z}_6$.

Theorem The group $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if $\gcd(m,n) = 1$.

Proof:

Show $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \Rightarrow \gcd(m,n) = 1$

Show that if $\gcd(m,n) = d > 1 \Rightarrow \mathbb{Z}_m \times \mathbb{Z}_n$ is not cyclic

$\Downarrow$

$m \mid \dfrac{mn}{d}, \quad n \mid \dfrac{mn}{d}$

$\therefore \underbrace{(a,b) + \cdots + (a,b)}_{\frac{mn}{d}} = 0$

$\frac{nm}{d} \cdot a = \left(\frac{n}{d}\right) \cdot ma = 0 \mod m$

$|\mathbb{Z}_m \times \mathbb{Z}_n| = mn$

$|(a,b)| \leq \frac{mn}{d}$ and $d > 1$ $\therefore$

$\langle (a,b) \rangle \neq \mathbb{Z}_m \times \mathbb{Z}_n$.

$\therefore \mathbb{Z}_m \times \mathbb{Z}_n$ is not cyclic.

$\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \Rightarrow \gcd(m,n) = 1$

If $\gcd(m,n) = 1$ then $|(1,1)| = \operatorname{lcm}(|\underset{\in \mathbb{Z}_m}{1}|, |\underset{\in \mathbb{Z}_n}{1}|)$

$= \operatorname{lcm}(m, n)$

$= mn$ - since relatively prime

$|(1,1)| = mn$ $\therefore$ $\mathbb{Z}_m \times \mathbb{Z}_n \cong \langle (1,1) \rangle$.

$\blacksquare$

Corollary 9.22. *Let $n_1, \ldots, n_k$ be positive integers. Then*

$$\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} = \prod_{i=1}^{k} \mathbb{Z}_{n_i} \cong \mathbb{Z}_{n_1 \cdots n_k}$$

if and only if $\gcd(n_i, n_j) = 1$ *for* $i \neq j$.

Corollary 9.23. *If*

$$m = p_1^{e_1} \cdots p_k^{e_k},$$

where the p_is are distinct primes, then

$$\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{e_1}} \times \cdots \times \mathbb{Z}_{p_k^{e_k}}.$$

$\gcd(p_i^{e_i}, p_j^{e_j}) = 1$ $i \neq j$

and p_k's prime.

(In chapter 13)

If G is finite, abelian

$G \cong \mathbb{Z}_{p_1^{a_1}} \times \cdots \times \mathbb{Z}_{p_k^{a_k}}$ ($p_i = p_j$ possibly)

Internal Direct Products

- External direct prod. builds a large group out of small groups

Break down a given group into the direct prod. of smaller groups

Def | Let G be a group, with subgroups H, K s.t.

- $G = HK = \{hk \mid h \in H, k \in K\}$
- $H \cap K = \{e\}$
- $hk = kh \quad \forall k \in K, h \in H$

G is the <u>internal direct product</u> of H and K.

Ex | $U(8)$ is the internal direct product

$\{1, 3, 5, 7\}$

$H = \{1, 3\}, \quad K = \{1, 5\}$

Ex | D_6 is the internal direct product of

$r^6 = 1$
$s^2 = 1$
$srs = r^{-1}$

$\underset{\simeq \mathbb{Z}_2}{H = \{1, r^3\}}, \quad \underset{\simeq S_3}{K = \{1, r^2, r^4, s, r^2 s, r^4 s\}}$

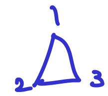
- $H \cap K = \{1\}$
- $HK = D_6$
- $hk = kh$

$\therefore D_6 \simeq \mathbb{Z}_2 \times S_3$

Ex] S_3 cannot be written as a interal direct product

$|S_3| = 6 \Rightarrow |H| = 3,\ |K| = 2$

The only subgroup of order 3 is



D_3

$$H = \{(1), (123), (132)\} = \{\text{id}, \text{rotatation}, \text{rotation}^2\}$$

But all subgroups of order 2 ie $K = \{(1), (12)\}$ are <u>not</u> s.t. $hk = kh\ \forall k \in K$ ∴ no internal direct product exists

Theorem 9.27. *Let G be the internal direct product of subgroups H and K. Then G is isomorphic to $H \times K$.*

<u>Proof</u>: G is an internal direct product $\Rightarrow \forall g \in G\ \ g = hk$, $h \in H$, $k \in K$

Define
$$\phi : G \longrightarrow H \times K$$
$$g \longmapsto (h, k)$$

Show ϕ is well defined. Need h, k to uniquely determine g

$g = hk = h'k'$

Consider

$$hk = h'k'$$

$$\cancel{h^{-1}h}\, \underbrace{k(k')^{-1}}_{\in K} = \underbrace{h^{-1}h'}_{\in H}\, \cancel{k'(k')^{-1}}$$

Since $H \cap K = \{e\}$

$$\underbrace{k(k')^{-1}}_{\in K} = \underbrace{h^{-1}h'}_{\in H} = e$$

$$h^{-1}h' = e = k(k')^{-1}$$

$$\Rightarrow h = h' \text{ and } k = k'$$

Show ϕ perserves group operation $g_1 = h_1 k_1$, $g_2 = h_2 k_2$

$$\phi(g_1 \cdot g_2) = \phi(h_1 k_1 h_2 k_2)$$

$$= \phi(h_1 h_2 k_1 k_2)$$ commutivity as a is interanl direct product

$$= (h_1 h_2, k_1 k_2)$$

$$= (h_1, k_1)(h_2, k_2) = \phi(g_1) \cdot \phi(g_2)$$

1-1 onto. Homework

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