Cosets and Lagrange's Theorem ← Helps us understand subgroups

Cosets:

Let G be a group, H a subgroup, $g \in G$.

Define a left coset of H with representative g as

$$gH = \{gh \mid h \in H\}$$

right coset

$$Hg = \{hg \mid h \in H\}$$

Ex) Let $H = \langle 3 \rangle = \{0, 3\}$ be the subgroup of $\mathbb{Z}_6$ gen. by 3

cosets are:

$$0 + H = 3 + H = \{0, 3\}$$
$$1 + H = 4 + H = \{1, 4\}$$
$$2 + H = 5 + H = \{2, 5\}$$

Ex) Let K be the subgroup of $S_3 = \{(1), (12), (13), (23), (123), (132)\}$ given by

$K = \{(1), (12)\}$

Left cosets

$$(1)K = (12)K = \{(1), (12)\}$$
$$(13)K = (123)K = \{(13), (123)\}$$
$$(23)K = (132)K = \{(23), (132)\}$$

Right cosets

$$K(1) = K(12) = \{(1), (12)\}$$
$$K(13) = K(132) = \{(13), (132)\}$$
$$K(23) = K(123) = \{(23), (123)\}$$

Lemma 6.3. *Let H be a subgroup of a group G and suppose that $g_1, g_2 \in G$. The following conditions are equivalent.*

1. $g_1 H = g_2 H$;
2. $H g_1^{-1} = H g_2^{-1}$;
3. $g_1 H \subseteq g_2 H$;
4. $g_2 \in g_1 H$;
5. $g_1^{-1} g_2 \in H$.

Theorem 6.4

Let H be a subgroup of a group G. Then the left (resp. right) cosets of H in G partition G.

That is G is the disjoint union of the left (resp. right) cosets of in H in G.

Proof:

Let $g_1 H$, $g_2 H$ to be two cosets of H in G, show that

$$g_1 H \cap g_2 H = \emptyset \quad \text{or} \quad g_1 H = g_2 H.$$

Suppose $g_1 H \cap g_2 H \neq \emptyset$ and $a \in g_1 H \cap g_2 H$ then

$$a = g_1 h_1 = g_2 h_2 \quad \text{for some } h_1, h_2 \in H$$

$$\Rightarrow \quad g_1 = g_2 \underbrace{h_2 h_1^{-1}}_{\in H} \Rightarrow g_1 \in g_2 H$$

$$g_1 = g_2 \tilde{h}$$

$$g_1 H = \{ g_1 h \mid h \in H \} = \{ g_2 \tilde{h} \cdot h \mid h \in H \}$$

$$= \{ g_2 \hat{h} \mid \hat{h} \in H \}$$

$$= g_2 H.$$

And All $g \in G$ appear in some coset, in particular in gH since $g \cdot e = g$ and $e \in H$. ∎

Def 1 Let G be a group, H is a subgroup. Define the index of H in G:

$$[G:H] = \# \text{ of left cosets of } H \text{ in } G$$

Ex 1 $G = \mathbb{Z}_6$, $H = \{0, 3\}$ $[G:H] = 3$

Theorem 6.8 Let H be a subgroup of G.

$\#$ left cosets of H in G = $\#$ right cosets of H in G.

Proof:

$\mathcal{L}_H$ - left cosets

$\mathcal{R}_H$ - right cosets

we wish to define a bijection $\phi : \mathcal{L}_H \to \mathcal{R}_H$.

If $gH \in \mathcal{L}_H$ let $\phi(gH) = Hg^{-1}$, note that this map is well defined since if $g_1H = g_2H = gH$ are diffent representitives of gH then $Hg_1^{-1} = Hg_2^{-1} = Hg^{-1}$ (by Lemma 6.3 part 1 and 2)

1-1: Suppose $\phi(g_1H) = \phi(g_2H)$

$$\Rightarrow Hg_1^{-1} = Hg_2^{-1} \Rightarrow g_1H = g_2H.$$

onto: For any $Hg \in \mathcal{R}_H$ we have that

$$\phi(g^{-1}H) = H(g^{-1})^{-1} = Hg \quad \therefore \phi \text{ is onto}$$

$\Rightarrow \phi$ is a bijection $|\mathcal{L}_H| = |\mathcal{R}_H|$.

Lagrange's Theorem

Thm: Let G be a finite group and let H be a subgroup of G.

Then $\frac{|G|}{|H|} = [G:H] =$ is the number of distinct left cosets of H in G.

In particular $|H| \mid |G|$

Proofs:

The group G is partioned (by Theorem 6.4) into $[G:H]$ distinct cosets

Each coset has $|H|$ elements

$\therefore \quad |G| = [G:H]\,|H|$ ∎

Corollary Suppose G is a finite group, $g \in G$

Then $|g| \mid |G|$. That is order of an element divides order of G

Proof: Apply Lagranges Theorem to $H = \langle g \rangle$.

Corollary Let $|G| = p$ for p prime. Then G is cyclic and is generated by any $g \in G$ s.t. $g \neq e$.

Proof:

Let $g \in G$, $g \neq e$.

Then $|g| \mid |G|$, but since $g \neq e$ $|g| > 1$ and $|g| \leq |G|$

$$\Rightarrow |g| = |G| \Rightarrow G = \langle g \rangle.$$ ∎

Corollary 1 Let H and K be subgroups of G, $|G| < \infty$ such that $K \subseteq H \subseteq G$. Then $[G:K] = [G:H][H:K]$.

Proof: From Lagranges Theorem

$$[G:K] = \frac{|G|}{|K|} = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = [G:H][H:K]. \blacksquare$$