Ideals in F[x]

Thrm $F[x]$ is a Principal ideal domain

Ex $F[x,y]$ is Not a PID

Consider

$I = \langle x, y\rangle$ in $F[x,y]$

$\langle x, y\rangle$ = all poly in $F[x,y]$ with no constant terms

$x \in I$, $y \in I$ but $\nexists$ no

$f(x,y)$ s.t. $x = f(x,y)\cdot y$ or

$y = f(x,y)\cdot x$

$$\therefore \langle x, y\rangle \neq \langle f(x,y)\rangle \quad \blacksquare$$

$F[x]/\langle p(x)\rangle$ is a feild.

Thm Let F be a feild and suppose $p(x) \in F[x]$

$I = \langle p(x)\rangle$ is maximal if and only if $p(x)$ is irreducible.

Proof

Suppose $I = \langle p(x)\rangle$ is maximal $\Rightarrow$ I is a prime ideal and a maximal ideal is proper

$$\therefore p(x) \neq 0.$$

Suppose $p(x) = f(x)g(x)$, $\deg(f) < \deg(p)$
$\deg(g) < \deg(p)$

$I = \langle p(x) \rangle$ is prime and $p(x) \in I \Rightarrow f(x)g(x) \in I$

$\therefore$ either $f(x) \in I$ or $g(x) \in I$.

Say its f. $\Rightarrow$ $f(x) = p(x)q(x) \Rightarrow \deg(f) \geq \deg(p)$

but this is a contradiction $\therefore$ $p(x)$ is irreducible

- Suppose $p(x)$ irr. over $F[x]$.

Say $\langle p(x) \rangle \subseteq I \subseteq F[x]$

I is a principal ideal since its an ideal of $F[x]$

$\therefore$ $I = \langle f(x) \rangle$ for some $f(x) \in F[x]$

$p(x) = f(x)g(x)$ for some $g \in F[x]$

But $p(x)$ is irreducible $\Rightarrow$ $f(x) = c \in F$ or $g(x) = c \in F$

- If $f(x) = c \Rightarrow I = \langle 1 \rangle = F[x]$

- If $g(x) = c \Rightarrow I = \langle p(x) \rangle = \langle f(x) \rangle$

$$f(x) = \frac{1}{c}p(x)$$

Ex) #38 ch 16

Idempotent $\iff$ $x^2 = x$ in R

If R is an int domain $\Rightarrow x = 0, 1$ are only ones:

$x = 0$ or $x \cdot x = x$

$x = 1$ if R is int. domain.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Ex) Find all ring hom $\phi: \mathbb{Z} \to \mathbb{Z}$.

Ans: $\phi(n) = 0$

or $\phi(n) = n$

Reason: $\phi(1) = \phi(1 \cdot 1) = \phi(1)\phi(1)$

$\phi(1) = \phi(1)^2 \Rightarrow \phi(1) = 0$ or 1

$$\phi(n) = \phi(\overbrace{1 + \cdots + 1}^{n \text{ times}}) = \overbrace{\phi(1) + \cdots + \phi(1)}^{n \text{ times}}$$

$= 0$ or n

Ex) Consider $f(x) = x^4 - 15$

- If f irr. in $\mathbb{Q}[x]$? Set $\alpha = (15)^{\frac{1}{4}}$

Roots of $f(x) = 0$ are $\pm\alpha, \pm i\alpha$

$\therefore$ no rational linear factors

Suppose

$$x^4 - 15 = (x^2 + ax + b)(x^2 + cx + d)$$

$$bd = -15$$

$$(x^2+ax+b)(x^2+cx+d) = x^4 + \overset{a=-c}{\underset{=0}{(a+c)}}x^3 + \underset{=0}{(b+d+ac)}x^2 + (bc+ad)x + bd.$$

$a = -c$

$b + d + ac = 0 \implies b + d - a^2 = 0$

$bc + ad = 0 \implies -ab + ad = 0$

$bd = -15$

$\underline{a(d-b) = 0}$

Either $a = 0 \implies c = 0$

$x^4 - 15 = (x^2+b)(x^2+d)$ but $b = -d$ (since $a=0$)

$= (x^2-b)(x^2+b)$

$\implies b^2 = 15$

$\implies b = \pm\sqrt{12} \notin \mathbb{Q}$

$\therefore a \neq 0 \implies b = d$ and $bd = -15$

$\implies b^2 = -15$

$\implies b \notin \mathbb{Q}$