Theorem) Let E be an extension field of F, $\alpha \in E$

α is trancendental over F if and only if

$F(\alpha) \simeq F(x)$

the feild of fractions of $F[x]$.

Proof: $\phi_\alpha : F[x] \to E$ ← eval. hom.

$f(x) \mapsto f(\alpha)$

By def., α is trancendental over F iff

$$\phi_\alpha(P(x)) = p(\alpha) \neq 0 \quad \forall \; p(x) \in F[x]$$

$\Rightarrow$ This is true iff $\ker \phi_\alpha = \{0\}$

i.e. iff ϕ_α is 1-1

$$F[x] \cong \underbrace{\phi_\alpha(F[x])}_{= F[\alpha]} \subset E$$

So E contains an isomorphic $F[x]$

the smallest field contains $F[x]$ is $F(x)$ ← feild of fractions

then, by Thm. 18.4, $F(x) \subset E$

$$\therefore \; F(\alpha) \cong F(x)$$

∎

Thm Let E be an extension feild of a feild F, $\alpha \in E$ algebraic over F. There exists a unique irruducable monic poly $f(x) \in F[x]$ of smallest degree s.t. $f(\alpha) = 0$. If $g(x) \in F[x]$, $g(\alpha) = 0$

$\Rightarrow f(x) \mid g(x)$

Proof: $\alpha \in E$ alg. over F

Let $\phi_\alpha : F[x] \to E$ (evaluation hom.)
$f(x) \to f(\alpha)$

$\ker \phi_\alpha = \langle f(x) \rangle$ with $\deg(f(x)) \geq 1$

$F[x]$ is a PID (since F is a field) and α alg. $\therefore$ $\exists$ poly of least degree s.t α is a root

so let $f(x)$ have least degree s.t. $f(\alpha) = 0$
(take $f(x)$ monic)

- $\langle f(x) \rangle$ consits exactly of all $g(x) \in F[x]$ s.t $g(\alpha)=0$

Since

- If $g(\alpha) = 0$, $g \neq 0 \Rightarrow g(x) \in \langle f(x) \rangle \therefore f(x) \mid g(x)$

and

$f(x)$ is irreducible since if $f(x) = r(x)s(x)$ for r, s, lower degree

then $f(\alpha) = r(\alpha)s(\alpha) = 0$

$\Rightarrow r(\alpha) = 0$ or $s(\alpha) = 0$

which is a contradiction since $f(x)$ has minimal degree s.t. α

is a root $\therefore$ $f(x)$ is irreducible.

Note we may take $f(x)$ monic since F is a feild.

Suppose $f(x) = \beta g(x)$ but if $g(x)$ is monic

then

$$x^n + a_{n-1}x^{n-1} + \cdots + = \beta\left(x^n + b_{n-1}x^{n-1} + \cdots\right)$$

$$\Rightarrow \quad \beta = 1$$

$\therefore$ $f(x) = g(x)$ $\therefore$ f is unique. $\blacksquare$

Def) Let E be an extension field of F, $\alpha \in E$ alg. over F. The unique monic polynomial $f(x)$ from ↑ theorem is called the minimal polynomial for α over F

$\deg(f(x)) =$ degree of α over F.

Ex) $x^2 - 2$ is the minimal poly. of $\sqrt{2}$ over $\mathbb{Q}$

$x^2 + 1$ is min poly of i over $\mathbb{Q}$ (or over $\mathbb{R}$)

$\sqrt{2}$, i have degree 2 over $\mathbb{Q}$.

Prop. Let E be a field extension of F, $\alpha \in E$ alg. over F. Then $F(\alpha) \simeq F[x]/\langle p(x)\rangle$ where $p(x)$ is the minimal polynomial of α over F.

Proof:

$$\phi_\alpha : F[x] \longrightarrow E \quad (\text{eval. hom})$$
$$f(x) \longmapsto f(\alpha)$$

$\ker(\phi_\alpha) = \langle p(x) \rangle$ where p is min. poly. of α.

By 1st iso theorem

$$F[x]/\langle p(x)\rangle \simeq \phi_\alpha(F[x]) \overset{F[\alpha]}{=}$$

↑ is a feild since $p(x)$ is irreducible

$\phi_\alpha(F[x]) = F(\alpha)$ since $\alpha \overset{\phi_\alpha(x)}{=} \in \phi_\alpha(F[x])$

and F (isomorphic copy) is in $\phi_\alpha(F[x])$

and $F(\alpha)$, by def. is the smallest feild with α and F.

and all we have is powers of α and elements of F. ∎

Theorem Let $E = F(\alpha)$ ← simple extension, $\alpha \in E$ algebraic over F

Suppose degree of α over F is n. Then every element $\beta \in E$ can be expressed uniquely in the form

$$\beta = b_0 + b_1\alpha + \cdots + b_{n-1}\alpha^{n-1}$$

for $b_i \in F$