**Prop.** Let $S = \{v_1, \dots, v_n\}$ be vectors in a v. space $V$

$\text{Span}_F(S)$ is a subspace of $V$.

## Linear Independece

**Def)** A set of vectors $v_1, \dots, v_n$ is <u>linearly indedendent</u>

if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

<u>Linearly depent</u> if there are non-zero $\alpha_i$'s s.t

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

**Prop)** Let $\{v_1, \dots, v_n\}$ be a linearly independent set in a v. space $V$.

Suppose $\alpha_1 v_1 + \cdots \alpha_n v_n = \beta_1 v_n + \cdots + \beta_n v_n$

$$\Rightarrow \quad \alpha_1 = \beta_1, \dots, \alpha_n = \beta_n$$

**Proof:**

$$\alpha_1 v_1 + \cdots \alpha_n v_n = \beta_1 v_1 + \cdots + \beta_n v_n$$

$$\Rightarrow \quad \underbrace{(\alpha_1 - \beta_1)}_{=0} v_1 + \cdots + \underbrace{(\alpha_n - \beta_n)}_{=0} v_n = 0$$

since $\{v_1, \dots, v_n\}$ are lin. indpedent. ∎

Prop) $\{v_1, \dots, v_n\}$ are linearly dependent iff one of the $v_i$'s is a linear combo. of the rest.

Prop) Suppose $V = \text{span}_F(v_1, \dots, v_n\}$ with $v_1, \dots, v_n$ lin. independent. If $m > n$ then any set of $m$ vectors in $V$ must be lin. dependent.

Def) $\{e_1, \dots, e_n\}$ is a basis of $V$ if $\{e_1, \dots, e_n\}$ is linearly independent and

$$V = \text{Span}_F(e_1, \dots, e_n)$$

Ex) $(1,0,0), \dots$ etc $\mathbb{R}^3$

Ex) $\{1, \sqrt{2}\}$, or $\{1+\sqrt{2}, 1-\sqrt{2}\}$ in $\mathbb{Q}(\sqrt{2})$

Prop) If $\{e_1, \dots, e_m\}$, $\{f_1, \dots, f_n\}$ are basis for $V$ then $m = n$.

Def) If $\{e_1, \dots, e_n\}$ is a basis for a v. space $V$

$\dim(V) = n$.

**Theorem 20.15.** *Let $V$ be a vector space of dimension $n$.*

1. *If $S = \{v_1, \ldots, v_n\}$ is a set of linearly independent vectors for $V$, then $S$ is a basis for $V$.*
2. *If $S = \{v_1, \ldots, v_n\}$ spans $V$, then $S$ is a basis for $V$.*
3. *If $S = \{v_1, \ldots, v_k\}$ is a set of linearly independent vectors for $V$ with $k < n$, then there exist vectors $v_{k+1}, \ldots, v_n$ such that*

$$\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$$

*is a basis for $V$.*

## Fields

- When is a feild $F$ contained in a lager feild?
- What feilds are between $\mathbb{Q}$ and $\mathbb{R}$?

Let $F$ be a feild, $p(x) \in F[x]$:

Can we find a feild $E$, $F \subseteq E$, s.t.

$p(x)$ factors into linear factors over $E[x]$?

Ex] consider $p(x) = x^4 - 5x^2 + 6 \in \mathbb{Q}[x]$

$$p(x) = (x^2 - 2)(x^2 - 3)$$

$\therefore$ $p$ has no zeros in $\mathbb{Q}$, has 4 zeros in $\mathbb{R}$

can find smaller feilds where $p(x)$ has zeros:

Extension feild

$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$

$$\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$$

- 2 roots in either feild.

# Extension Fields

A feild $E$ is an <u>extension field</u> of a field $F$ if $F$ is a subfeild of $E$. $F$ is called the base of $E$. write $F \subset E$.

Ex) $F = \mathbb{Q}(\sqrt{2}) = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \}$

$E = \mathbb{Q}(\sqrt{2} + \sqrt{3}) = \{ a + b(\sqrt{2} + \sqrt{3}) \mid a, b \in \mathbb{Q} \}$

$E$ is an extension feild of $F$:

$\sqrt{2} + \sqrt{3} \in E \quad \therefore \quad \dfrac{1}{\sqrt{2} + \sqrt{3}} = \sqrt{3} - \sqrt{2} \in E$

Note we now have an extension feild with all roots of $x^4 - 5x^2 + 6$

Aside)

feild of fractions of $\mathbb{Z}[\sqrt{n}] \cong \mathbb{Q}(\sqrt{n})$

( Home work problem )

any element of feild of fractions of $\mathbb{Z}[\sqrt{n}]$

$\dfrac{c + d\sqrt{n}}{e + f\sqrt{n}} \longrightarrow a + b\sqrt{n}$

Do conjugation, simplify.