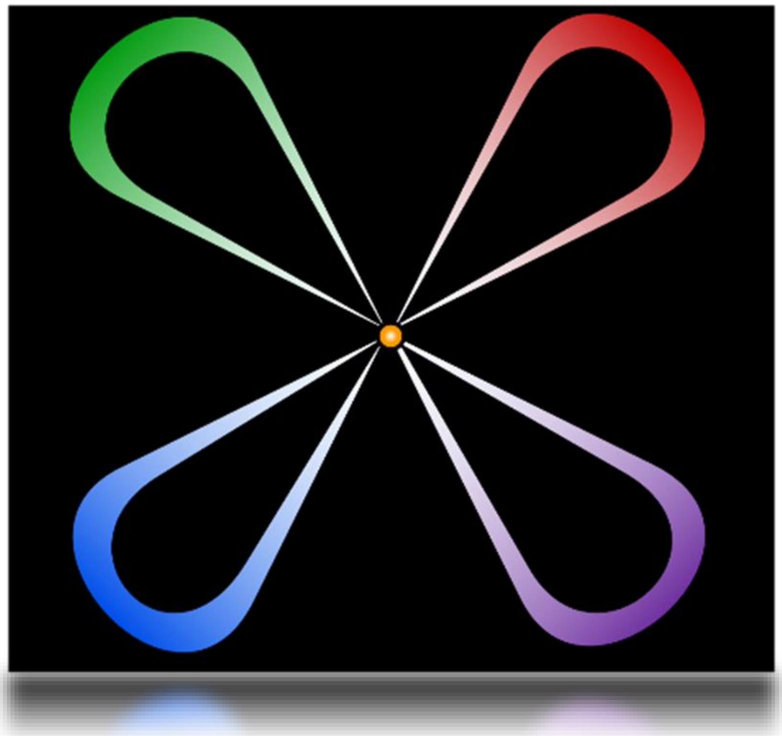
STRATIFIED INVARIANTS for KINEMATIC SINGULARITIES



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ABSTRACT

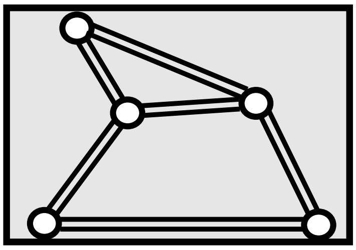
Motion planning and optimal control form substantial endeavors across theoretical and practical aspects of research and development in inverse kinematics. Both are hindered by **singularities**, i.e., those regions of configuration space where the mechanism at hand loses at least one degree of freedom. Equivalently, singularities are those regions where the derivative of the kinematic map fails to attain full rank. The prevailing wisdom in kinematics research is that such singularities should be quickly identified and steadfastly avoided.

Here we suggest a fundamentally different approach: *a deeper geometric and topological understanding of singularities will be enormously beneficial* to inverse kinematics. The ability to efficiently probe the fibers of a kinematic map at (or even near) a singularity allows us to properly catalog safe directions as well as prohibited ones in the configuration space. The knowledge of such directions will, in turn, facilitate far more efficient solutions to standard motion planning problems by removing superfluous constraints: we could safely pass through singularities rather than forcing ourselves to bypass them.

For a wide and practically relevant class of kinematic maps, we aim to algorithmically decompose the configuration and state spaces into sub-manifolds so that the kinematic map in question (a) sends sub-manifold to sub-manifold, and (b) has a constant rank derivative when restricted to each sub-manifold. This **stratification** serves as an essential starting point for a thorough analysis of various kinematic singularities, their computable invariants, and hence, of their mechanical passability.

INTRODUCTION

The kinematic map associated to a robot is a function $f: C \to S$ where C and S denote the input *configuration space* and the output *state space* of that robot respectively. The **motion planning** problem [1] amounts to finding an optimal path of the form $\gamma: [0,1] \to C$ interpolating from the initial state $f \circ \gamma(0)$ to some desired target state $f \circ \gamma(1)$. It is standard practice to assume that both C and S are smooth sub-manifolds of Euclidean space and that f is a smooth mapping between them. The key advantage of imposing smoothness is that well-known tools from differential calculus and linear algebra may be brought to bear on the vital task of finding optimal trajectories in configuration space which achieve desired target states.



THE CHALLENGE: Even when f, C and S are all smooth, the mapping f may still admit intricate **singularities** as follows. Writing m and n to denote the dimensions of C and S respectively, the *singular locus* of f is the subset $\Sigma_f \subset C$ consisting of all those points p where the derivative $df_p: T_pC \to T_{f(p)}S$ between tangent spaces, which is represented by an $n \times m$ Jacobian matrix of partial derivatives evaluated at p, has rank strictly smaller than $\min(m, n)$. These singularities are generic in the sense that Σ_f is non-empty for any reasonably complicated f. Much of the traditional differential calculus toolbox fails at Σ_f-points, since iterative optimization schemes such as gradient descent require df_p to admit at least a one-sided inverse whenever p lies along a gradient-minimizing trajectory.

The typical remedy involves imposing Σ_f-avoidance as an additional constraint when solving for optimal paths. This is often ensured in practice by *over-actuation*, i.e., by concocting higher-dimensional configuration spaces to enforce $m \gg n$. One then hopes to find enough redundancy in the fiber $f^{-1}(q) \subset C$ of a given state $q \in S$ that any singularities in sight may be bypassed via a *null move*, i.e., a path lying entirely within this fiber [2]. There are two unfortunate by-products of this strategy. First, higher-dimensional configuration spaces naturally incur much higher computational costs from an algorithmic perspective. Second, and far more serious, is the fact that not all singularities are created equal. While some are indeed unsafe, others can be effortlessly traversed in certain directions. Thus, forcing an optimizer to avoid Σ_f entirely might overlook genuinely optimal and mechanically safe paths.

OUR MOTIVATION: We seek to improve on this state of affairs by constructing a toolbox for thorough local and global analysis of the singular locus Σ_f whenever the kinematic map f is given in terms of polynomial or trigonometric polynomial equations. The first step is to algorithmically decompose $\Sigma_f \subset C$ and its image $f(\Sigma_f) \subset S$ into manifold pieces, of possibly different dimensions, so that the restriction of f to each sub-manifold $M \subset \Sigma_f$ maps onto a sub-manifold $N \subset f(\Sigma_f)$ with derivative $df_p: T_pM \to T_{f(p)}N$ of constant rank across all $p \in M$. This is called a **stratification** of f subordinate to its *Thom-Boardman flag*. Having an efficient algorithm to construct such a stratification opens the door to a host of useful tools from geometry and topology being deployed to aid with motion planning and optimization through the singular locus. Moreover, with this machinery in hand, we are able to treat singular and non-singular spaces on an equal footing; in particular, we allow non-manifold configuration and state spaces.

APPROACH

We will assume that the configuration space $C \subset \mathbb{R}^d$ and state space $S \subset \mathbb{R}^e$ are *real algebraic varieties*, i.e., subsets of Euclidean space given as zero sets of polynomials[1], while the kinematic map $f: C \to S$ is the restriction of a polynomial function $F: \mathbb{R}^d \to \mathbb{R}^e$. Letting $k \leq \min(d, e)$ be the largest rank attained by the derivative dF on $\mathbb{R}^d$, the **Thom-Boardman flag** [3] of F is an increasing sequence $V_0 \subset V_1 \subset \cdots \subset V_k = \mathbb{R}^d$ of sub-varieties defined as follows: V_i consists of all those points $p \in \mathbb{R}^d$ for which the rank of dF_p is $\leq i$. Intersecting the V_i with C produces the Thom-Boardman flag of f, with the singular locus being given by $\Sigma_f = V_{k-1} \cap C$. And similarly, the images $U_i = f(V_i \cap C)$ form a nested sequence of sub-varieties of S, with $f(\Sigma_f) = U_{k-1}$.

[1] All instances of "polynomial" here and elsewhere may be safely replaced by "trigonometric polynomial."

SETUP: Real algebraic varieties are rarely smooth manifolds: the picture on the title page, for instance, is the zero set of $(x^4 + y^4 + x^2 + y^2)^2 - 9x^2y^2$, and the origin is evidently a non-manifold point. Nevertheless, in [4], H. Whitney established that every such variety can be decomposed into finitely many smooth manifold pieces of possibly different dimensions, usually called *strata*, which fit together coherently in the sense that small neighbourhoods around two points in the same stratum are geometrically identical. Moreover, the closures of these strata are algebraic sub-varieties and hence also admit a finite description as the vanishing locus of certain polynomials. These decompositions into strata are called *Whitney stratifications*. Here is the key object of interest for our purposes.

DEFINITION: A **Thom-Boardman subordinate stratification** of $f: C \to S$ is given by Whitney stratifications of C and S subject to the following requirements:

1. Every stratum $M \subset C$ lies in a single successive difference $V_i - V_{i-1}$.
2. Every stratum $N \subset S$ lies in a single successive difference $U_j - U_{j-1}$.
3. The image $f(M)$ of a stratum $M \subset C$ is wholly contained in a stratum $N \subset S$.
4. At each point $p \in M$, the derivative $df_p: T_pM \to T_{f(p)}N$ is surjective.

It follows from these requirements that for each stratum $N \subset S$, the restriction $f|_{f^{-1}(N)}$ is a *fiber bundle* [5] of the form $f^{-1}(N) \to N$; and as a consequence, the fibers $f^{-1}(q)$ and $f^{-1}(q')$ share important geometric and topological properties whenever both q and q' belong to the same stratum N. In particular, the fiber-wise homology groups are identical. The ranks of these groups, often called *Betti numbers*, play a crucial role in deciding whether or not a singularity is safe [2].

WORK PLAN: The starting point of the proposed work is our recent paper [6], where we have described the first practical algorithm for constructing Thom-Boardman subordinate stratifications of algebraic maps between *complex* algebraic varieties. This paper also contains the first practical algorithm for constructing Whitney stratifications of such varieties, which we have implemented in the Macaulay2 language [7]. All other known approaches involve some variant of the cylindrical algebraic decomposition (CAD) algorithm [8], which is prohibitively expensive to compute in practice. While our algorithm bypasses CAD entirely through the use of *conormal maps* and Gröbner basis techniques (that are specific to complex varieties), we have collected compelling preliminary evidence suggesting that the same methods may also be used to stratify real varieties. Over the course of five years, we will build tools to effectively study the singularities of algebraic maps between real varieties.

YEAR 1: We will extend the conormal space-based Whitney stratification algorithm to work for real algebraic varieties (such as C and S) and the Thom-Boardman subordinate stratification to real algebraic maps (such as $f: C \to S$); this will include prototype implementation code as well as theoretical guarantees of correctness. Towards the end of the year, we will also recruit a postdoc at NC State to work with us on this project for the next two years.

YEAR 2: There is no known effective and practical-to-implement algorithm for computing the Betti numbers of fibers of the form $f^{-1}(q)$ for a given state $q \in S$ directly from the defining equations of f, C and S.[2] We will establish a sampling theorem to learn the Betti numbers of such fibers from sufficiently large point samples using techniques from topological data analysis. The PIs have significant experience with such *homological inference* results [9, 10, 11].

YEAR 3: For the purposes of efficient optimisation, is it not enough to only know the Betti numbers of the fibers of f, one also requires knowledge of the geometry of the singular locus Σ_f and its image $f(\Sigma_f)$. We will establish homological inference theorems for these two spaces by making use of their Whitney stratifications described above; we will also take the opportunity to identify areas of theoretical and computational improvement in our stratification and inference algorithms.

[2] There are only finitely many computations to perform since we require at most one q from each stratum $N \subset S$.

YEAR 4: We will use knowledge of fiber-wise Betti numbers to identify precisely when a gradient descent trajectory is allowed to safely pass from one stratum of the configuration space C to another, thus augmenting the space of candidate optimal paths to include those which safely permeate the singular locus (instead of avoiding it altogether). We will also explore the computation of other stratified invariants of f, including its *local Euler obstruction*, its action on *Chern-Schwartz-MacPherson* classes, and its *characteristic cycles*, to see whether these perform any better than fiber-wise Betti numbers for classifying kinematic singularities. The PIs have extensive experience with such computations [11, 12, 13].

YEAR 5: Here we will explore kinematic problems without any recourse to defining equations. In other words, the only knowledge we assume of C and S are finite point samples, and f will only be known as a correspondence between these sample points. We hope to establish analogues of the stratification and homology inference results proved in previous years to this much more difficult setting.

PAYOFF

The most immediate benefit of this project occurs already in Year 1; by the end of this year, we will be able to replace infinitely many pointwise fiber Betti number computations by finitely many strata-wise ones. By the end of Year 2, we will be able to classify singularities in Σ_f as passable or not with high confidence. By the end of Year 3, we will have a framework for optimal control that is unconstrained by superfluous Σ_f-avoidance. At the end of Year 4, we will generate comprehensive benchmarks comparing stratified invariants in terms of how well they are able to classify kinematic singularities. Our most wide-ranging contribution would occur at the end of Year 5, where we will be able to extract actionable intelligence from black-box robots with no prior knowledge of their governing principles.

This research will broadly benefit researchers in mathematics as well as kinematics and computer science. All theoretical and computational results will be submitted for publication in high-impact international journals, such as Advances in Mathematics, Foundations of Computational Mathematics, International Mathematics Research Notices, and so forth. All code generated as part of this project will be made freely available under a permissive (GPL) license.

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