Segre-Driven Methods to find Algebraic Multiplicity & Test Ideal Membership

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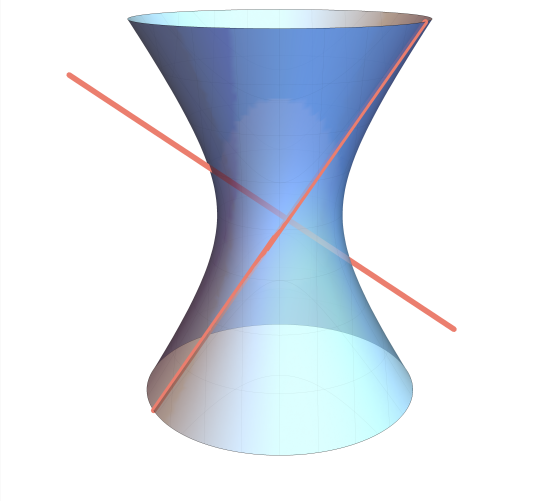
September 18, 2018



Goals and Setting

Segre classes figure prominently in Fulton-MacPherson Intersection Theory, where they are seen to contain a wide variety of geometric and enumerative information regarding pairs of varieties (or schemes).

However using the information encoded in Segre classes to answer classical computational questions has previously been impractical.

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However using the information encoded in Segre classes to answer classical computational questions has previously been impractical.

We will discuss a new algorithm to compute Segre classes that leads to several practical computational methods which can:

- Test containment of varieties (i.e. prime ideal membership) without computing a Gröebner basis.
- Test if a variety X is contained in the singular locus another variety Y without finding the ideal of the singular locus of Y.
- Compute the algebraic multiplicity of subvarieties without working in local rings.

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The Segre class $s(X, Y)$ is a function of two varieties (or schemes) $X \subset Y$. We will focus on the case, $X \subset Y \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ (all results still hold if $X \subset Y \subset T_\Sigma$ for any smooth projective toric variety T_Σ).

Here Segre classes are polynomials in the *Chow ring*:

$$A^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}) \cong \mathbb{Z}[h_1, \ldots, h_m]/(h_1^{n_1+1}, \ldots, h_m^{n_m+1})$$

The h_i represents the class of a general hyperplane in the ith factor $\mathbb{P}^{n_i}$.

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Example

Work in $\mathbb{P}^2_x \times \mathbb{P}^3_y$ over $\mathbb{C}$ (or any algebraically closed field) and consider the hypersurface $Y = V(x_1^2 x_0 y_3^2 - x_0^3 y_2 y_0 - x_0^3 y_0^2)$. This hypersurface has multidegree $(3, 2)$ and its class is

$$[Y] = 3h_1 + 2h_2 \in A^*(\mathbb{P}^2 \times \mathbb{P}^3) \cong \mathbb{Z}[h_1, h_2]/(h_1^3, h_2^4).$$

Let $X = V(x_0^2 x_1 y_1^2 - x_0^3 y_0 y_3, x_1^2 x_0 y_3^2 - x_0^3 y_2 y_0 - x_0^3 y_0^2)$, then $X \subset Y$ and

$$s(X, Y) = -624h_1^2h_2^3 + 180h_1^2h_2^2 + 88h_1h_2^3 - 42h_1^2h_2 - 32h_1h_2^2 - 8h_2^3 + 6h_1^2 + 10h_1h_2 + 4h_2^2 + h_1$$

Segre Classes

Segre classes can be used to find many interesting things:

- Chern-Schwartz-MacPherson (c_{SM}) class
- Euler characteristic
- Local Euler obstructions
- Samuel's multiplicity of a subvariety
- Test if two ideals have the same integral closure
- Chern-Mather (c_M) class
- Euclidean distance degree
- polar degrees
- Fulton & MacPherson's intersection product in arbitrary projective varieties
- Containment of varieties

Segre classes have historically been difficult (or impossible) to compute. The first practical methods to compute Segre classes were due to Aluffi (2003) in the special case $s(X, \mathbb{P}^n)$, these were improved by Eklund, Jost, and Peterson (2011) and Helmer (2014). These methods were extended to the case $s(X, Y)$ for X, Y subvarieties of $\mathbb{P}^n$ by Harris (2015). Our current work generalizes all previous methods, opening the way for new applications.

Projective Degrees

Let $X \subset Y \subset \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ be closed subschemes, choose generators of the ideal $\mathcal{I}_X = (f_0, \ldots, f_r)$ defining X so that all have the same multidegree.

The *projection of Y from X* is the rational map:

$$pr_X : Y \dashrightarrow \mathbb{P}^r, \quad pr_X(p) = (f_0(p) : \cdots : f_r(p)).$$

The projective degrees of X in Y, $g_a(X, Y)$, are

$$\sum_{i=0}^{\dim(Y)} \left[\overline{pr_X^{-1}(\mathbb{P}^{r-(\dim(Y)-i)}) - X} \right] = \sum_{|a|=a_1+\cdots+a_m \leq \dim(Y)} g_a(X, Y) \cdot h^{n-a} \in A^*(\mathbb{P}),$$

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Projective degrees measure how algebraically dependent $f_0, \ldots, f_r$ are in Y.

Proposition (Harris & Helmer, 2018)

The projective degrees of X in Y are given by

$$g_a(X, Y) = \deg\left(Y \cap L^a \cap W - X\right),$$

where $W = \mathbb{V}(P_1, \ldots, P_{\dim(Y)-|a|})$, with $P_j = \sum_i \lambda_i f_i$ for general $\lambda_i \in \mathbb{C}$ and L^a is a general linear space with class $h^a = h_1^{a_1} \cdots h_m^{a_m}$.

Effective Formulas for Segre Classes in Multi-Projective Space

Let $X \subset Y \subset \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ be closed subschemes and let $(d_1, \ldots, d_m)$ be the (component wise) maximum multidegree of a set of generators for the ideal $\mathcal{I}_X$ defining X.

Theorem (Harris & Helmer, 2018)

The Segre class $s(X, Y) \in A^*(\mathbb{P})$ is an **explicitly given** *function of the projective degrees $g_a(X, Y)$ for $|a| = \dim(X), \ldots, 0$ and $(d_1, \ldots, d_m)$, the max multidegree of the generators of $\mathcal{I}_X$.*

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Corollary

The dimension X part of the Segre class, $\{s(X, Y)\}_{\dim(X)}$, is completely determined by $(d_1, \ldots, d_m)$ and the projective degrees $g_a(X, Y)$ in dimension X, i.e. where $|a| = \dim(X)$.

Segre Classes and Samuel's Algebraic Multiplicity

Let R be the multigraded coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$, this ring has a $\mathbb{Z}^m$ grading with the first $n_1 + 1$ variables graded by $(1, 0, \ldots, 0)$, and so on.

Let I be a prime ideal in R defining a variety X and let $I \supset J$ be a primary ideal in R, defining a scheme Y. The local ring of Y along X is defined as the localization of R/J at the prime ideal I, that is

$$\mathcal{O}_{X,Y} = (R/J)_I.$$

Let $\mathcal{M}$ denote the maximal ideal of $\mathcal{O}_{X,Y}$. For $t >> 0$ and $d = \operatorname{codim}(X, Y)$ the Hilbert-Samuel polynomial is

$$P_{HS}(t) = \ell(\mathcal{O}_{X,Y}/\mathcal{M}^t) = e_X Y \cdot \frac{t^d}{d!} + \text{lower terms}.$$

The coefficient $e_X Y$ of $\frac{t^d}{d!}$ is the *algebraic multiplicity* of X on Y.

Segre Classes and Samuel's Algebraic Multiplicity

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Example (Twisted Cubic V1)

$R = \mathbb{C}[x, y, z, w]$ is the coordinate ring of $\mathbb{P}^3$, $A^*(\mathbb{P}^3) \cong \mathbb{Z}[h]/(h^4)$.

Define the prime ideal $I = (yw - z^2, xw - yz, xz - y^2)$

and the primary ideal $J = (z(yw - z^2) - w(xw - yz), xz - y^2)$, $J \subset I$.

Let $X = V(I)$, Y the scheme associated to J and note that $[X] = 3h^2$. We have

$$s(X, Y) = 6h^2 = 2[X] \in A^*(\mathbb{P}^3)$$

which gives that $e_X Y = 2$.

Segre Classes and Samuel's Algebraic Multiplicity

Theorem (Harris & Helmer, 2018)

Let $Y \subset \mathbb{P}^n$ be a pure-dimensional subscheme and $X \subset Y$ a non-empty subvariety. Let d be the maximum degree of the generators of $\mathcal{I}_X$ and $\mathcal{I}_Y$, and set $C = \dim(Y) - \dim(X)$. Then

$$e_X Y = \frac{\deg(Y) d^C - g}{\deg(X)} \in \mathbb{Z}_{>0},$$

where g is the $\dim(X)$ projective degree of X in Y.

A similar result holds in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$.

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Example (Twisted Cubic V2)

Again X is the twisted cubic and Y the 'doubly twisted' cubic. We have

$$e_X Y = \frac{\deg(Y) d^{\dim(Y)-\dim(X)} - g}{\deg(X)} = \frac{6 \cdot 3^0 - 0}{3} = 2.$$

Containment in the Singular Locus

Proposition (Samuel, 1955)

Let $X \subseteq Y$ be subvarieties of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. Then $e_X Y = 1$ if and only if X is not contained in the singular locus of Y.

Example (Containment of Varieties)

Work in $\mathbb{P}^6_x$, let $X = \mathbb{V}(x_0, x_1, x_2, x_3, x_4) = \mathbb{V}(K)$ and let $Y = \mathbb{V}(I)$ be an irreducible singular surface of degree 20 in $\mathbb{P}^6$ with

$$I = 3\times 3 \text{ minors of } \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 3x_3 & 4x_4 & 5x_5 & 6x_6 \\ x_2 & x_3 & x_4 & x_5 \\ x_0 + 5x_1 & x_1 + 6x_2 & x_2 + 7x_3 & x_3 + 8x_4 \end{pmatrix}.$$

Is X contained in $\mathrm{Sing}(Y)$?

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Yes, since $e_X Y = \frac{20 \cdot 3^{2-1} - g}{1} = \frac{60-58}{1} = 2 > 1$.

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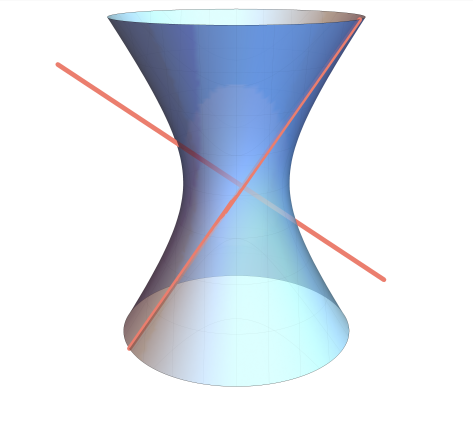
Yes, since $e_X Y = \frac{20 \cdot 3^{2-1} - g}{1} = \frac{60-58}{1} = 2 > 1$. This takes 0.07s.

Computing minors to get ideal J defining $\mathrm{Sing}(Y)$ takes 690s, reducing the generators of J w.r.t. K takes another 2s.

Segre Classes and Geometric Containment

Let X and Y be arbitrary nonempty subschemes of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ defined by ideals $\mathcal{I}_X = (f_1, \ldots, f_r)$ and $\mathcal{I}_Y = (\omega_1, \ldots, \omega_s)$ (where all generators are chosen to have the same multidegree).



Let $\sum \lambda_i f_i$ and $\sum \gamma_j \omega_j$ be general $\mathbb{C}$-linear combinations defining hypersurfaces Θ_X and Θ_Y, respectively.

Theorem (Harris & Helmer, 2018)

Let $Z = \Theta_X \cup \Theta_Y$. Then a top-dimensional irreducible component V of X is contained in Y if and only if

$$\{s(X, Z)\}_{\dim(X)} \neq \{s(X, \Theta_X)\}_{\dim(X)}.$$

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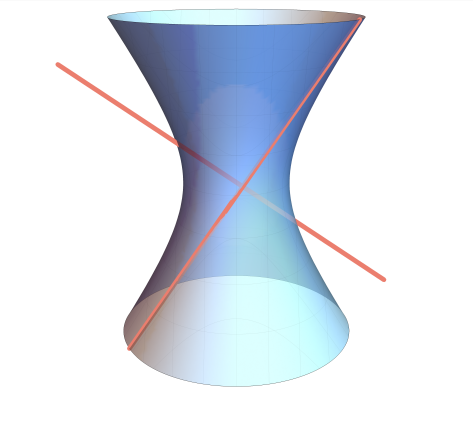

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This solves ideal membership for *prime* ideals without a Gröbner basis.

Example (Containment in $\mathbb{P}^2_x \times \mathbb{P}^2_y \times \mathbb{P}^2_z$)

The irrelevant ideal is $B = (x_0, x_1, x_2) \cdot (y_0, y_1, y_2) \cdot (z_0, z_1, z_2)$. Let F be a general polynomial of multidegree $(1, 1, 1)$ and let

$$f_1 = -10x_2y_1z_0 + 2x_1y_2z_0 + 35x_2y_0z_1 - 7x_0y_2z_1 - 25x_1y_0z_2 + 25x_0y_1z_2$$

$$f_2 = 9x_2y_1z_0 - 9x_1y_2z_0 - 4x_2y_0z_1 + 4x_0y_2z_1 + 3x_1y_0z_2 - 3x_0y_1z_2.$$

Consider $X = \mathbb{V}(\mathcal{I}_X)$, $Y = \mathbb{V}(\mathcal{I}_Y)$ where $\mathcal{I}_X = (y_0, y_1, y_2) \cdot (f_1, f_2) + (F)$, and $\mathcal{I}_Y = (z_0 \cdot f_1 - z_1 \cdot f_2, F)$.

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Is the irreducible variety X contained in Y?

Yes, since $\{s(X, \Theta_X \cup \Theta_Y)\}_{\dim(X)} = 2[X] \neq [X] = \{s(X, \Theta_X)\}_{\dim(X)}$.

Θ_X is defined by a general linear combination of a multidegree $(1, 2, 2)$ set of gens for $\mathcal{I}_X$, Θ_Y is defined by a general linear combination of multidegree $(1, 1, 2)$ gens for $\mathcal{I}_Y$.

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This takes 2.3s. Verifying $\mathcal{I}_Y : B^\infty \subset \mathcal{I}_X : B^\infty$ using Gröbner bases takes 11.4s.

Example (Equality of Irreducible Components (or radical ideals))

Work in $\mathbb{P}^6_x$, let $f_1 = x_2x_3x_5 - 5x_6^2x_0 + 3x_2x_0x_1$ and let f_2 be a general poly. of degree three. Consider the variety X defined by $\mathcal{I}_X = (f_1, f_2)$, and the irreducible scheme Y defined by $\mathcal{I}_Y = (f_1^2, f_1f_2, f_2^2)$.

We wish to verify that $\mathcal{I}_X = \sqrt{\mathcal{I}_Y}$.

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$$Z = \mathbb{V}(\lambda_1 f_1 + \lambda_2 f_2) \cup \mathbb{V}(\lambda_3 f_1^2 + \lambda_4 f_1 f_2 + \lambda_5 f_2^2) = \Theta_X \cup \Theta_Y \text{ for general } \lambda_i \in \mathbb{C}.$$

Computing the appropriate dim. X (res. Y) projective degrees we obtain:

$$\{s(X, Z)\}_{\dim(X)} = 9[X] \neq 3[X] = \{s(X, \Theta_X)\}_{\dim(X)},$$
$$\{s(Y, Z)\}_{\dim(X)} = 6[Y] \neq 4[Y] = \{s(Y, \Theta_Y)\}_{\dim(X)}$$

Hence $X \subset Y$, $Y_{\text{red}} \subset X \Rightarrow X = Y_{\text{red}}$, and $\mathcal{I}_X = \sqrt{\mathcal{I}_Y}$ (takes 2.4s).

We used the fact that X and Y are irreducible. If we had only known X was a variety $\Rightarrow$ an irreducible component of Y was equal to X.

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Note we use $\mathcal{I}_X$ and $\mathcal{I}_Y$ only; we *do not* compute $\sqrt{\mathcal{I}_Y}$.

Computing $\sqrt{\mathcal{I}_Y}$ to verify $\mathcal{I}_X = \sqrt{\mathcal{I}_Y}$ takes > 8 hours.

(More) Applications of Segre Classes: Intersection Products

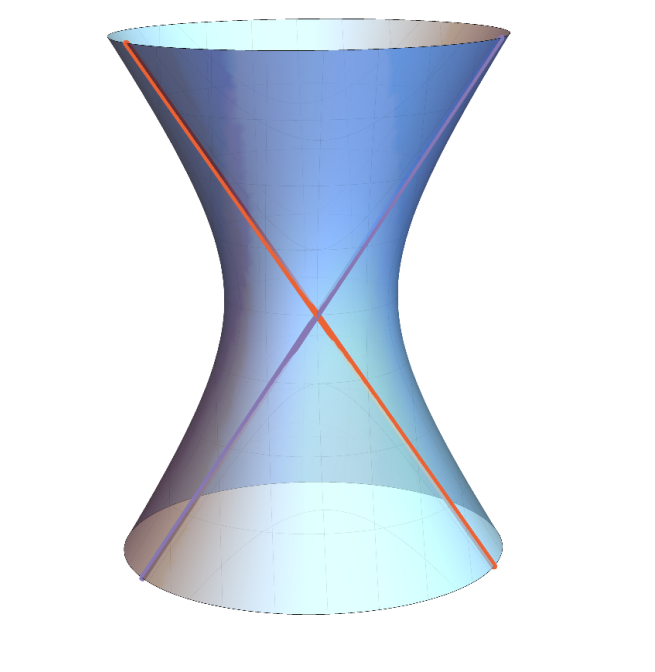
Goal: Understand the intersection of two projective varieties inside another, i.e. what does $X \cap V$ look line inside Y.

The intersection product $X \cdot_Y V$ lets us do this.

Example (Lines in a Quadric)

Suppose that $\mathbf{L}_1$ and $\mathbf{L}_2$ are lines from the families of orange and purple lines (respectively) on the blue quadric Q in $\mathbb{P}^3$.

The intersection products $\mathbf{L}_1 \cdot_Q \mathbf{L}_1$ and $\mathbf{L}_1 \cdot_Q \mathbf{L}_2$ capture the behavior of the intersections of two orange lines and a orange and a purple line (respectively) *inside* the quadric Q.



(More) Applications of Segre Classes: Intersection Products

Let $Y \subset \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ be a nonsingular variety, let X be regularly embedded in Y and let V by any subvariety of Y.

Fulton & MacPherson define the *intersection product of V by X in Y* as

$$X \cdot_Y V = \{c(N) \frown s(X \cap V, V)\}_{\text{expected dim.}} \in A^*(Y).$$

N denotes the (pull back to X of the) *normal bundle* $N_X Y$ (the vector bundle on X with sections dual to I_X/I_X^2).

The product $X \cdot_Y V$ captures the behavior of the intersection $X \cap V$ *inside* of the variety Y.

$X \cdot_Y V$ is also the product of $[X]$ and $[V]$ in the Chow ring $A^*(Y)$.

It is often impossible to obtain an explicit form for $A^*(Y)$. Using Segre classes we can compute $X \cdot_Y V$ without knowing $A^*(Y)$.

(More) Applications of Segre Classes: Intersection Products

Example (Lines in a Quadric)

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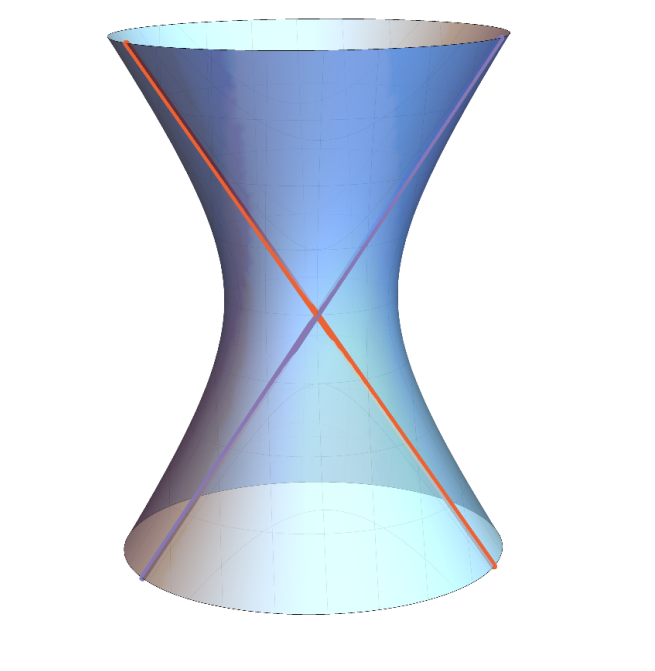

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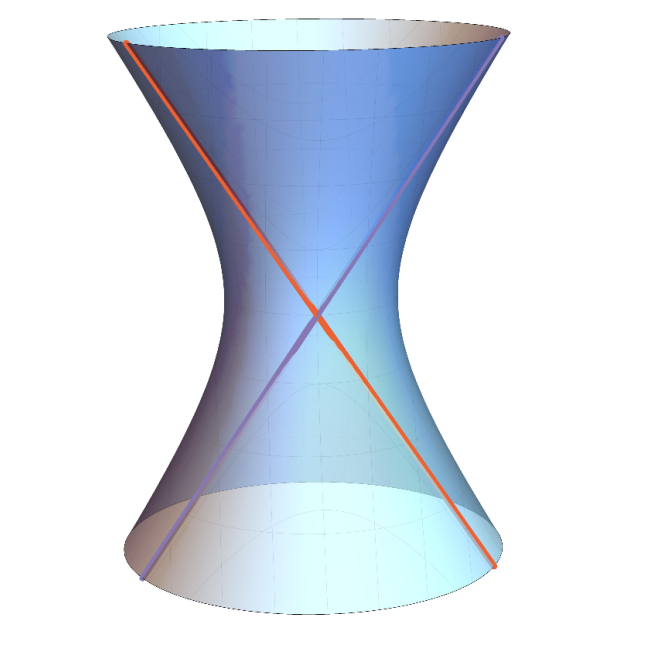
Work in $A^*(\mathbb{P}^3) \cong \mathbb{Z}[h]/(h^4)$, $A^*(\mathbb{P}^3 \times \mathbb{P}^3) \cong \mathbb{Z}[h_1, h_2]/(h_1^4, h_2^4)$.



$$\mathbf{L_1} \cdot_Q \mathbf{L_1} = \{c(T_Q) \frown \Delta^*(s(L_1 \cap L_1, Q \times Q))\}_{\text{dim. } 0}$$

$$= \{c(T_Q) \frown \Delta^*(h_1^3 h_2^2 + h_1^2 h_2^3 - 2h_1^3 h_2^3)\}^{(\nu)} = \left\{ \frac{(1+h)^4}{1+2h} \cdot (h^2 - 2h^3) \right\}_{\text{dim. } 0} = 0 = \text{empty}$$

$$\mathbf{L_1} \cdot_Q \mathbf{L_2} = \{c(T_Q) \frown \Delta^*(s(L_1 \cap L_2, Q \times Q))\}_{\text{dim. } 0}$$

$$= \{c(T_Q) \frown \Delta^*(h_1^3 h_2^3)\}^{(\nu)} = \left\{ \frac{(1+h)^4}{1+2h} \cdot h^3 \right\}_{\text{dim. } 0} = h^3 = \text{one point}$$

Computing Projective Degrees Revisited

Let $X \subset Y \subset \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ be closed subschemes and choose generators of the ideal $\mathcal{I}_X = (f_0, \ldots, f_r) \subset R = \mathbb{C}[x^{(1)}, \ldots, x^{(m)}]$ defining X so that all have the same multidegree. Recall that the projective degrees of X in Y are $g_a(X, Y) = \deg\left(Y \cap L^a \cap W - X\right)$, where

$$W = \mathbb{V}(P_1, \ldots, P_{\dim(Y)-|a|}), \quad P_j = \sum_i \lambda_i f_i \ \text{ for general } \ \lambda_i \in \mathbb{C}.$$

One could compute this is several ways, in practice we use the method below.

Theorem (Harris & Helmer, 2018)

The projective degrees of X in Y are given by

$$g_a(X, Y) = \dim_{\mathbb{C}} \frac{R[T]}{\mathcal{I}_Y + \mathcal{I}_{L^a} + A + (P_1, \ldots, P_{\dim(Y)-|a|}, 1 - T \cdot P_0)},$$

where $A = (\ell(x^{(1)}) - 1, \ldots, \ell(x^{(m)}) - 1)$ for $\ell(x^{(i)})$ a general linear form in $x^{(i)}$ and L^a is a general linear space with class h^a.

Implementation

The main computational steps of all our methods is to find the number of solutions to zero dimensional polynomial systems. The test implementation uses M2's F4 GB; numeric methods could be used.

http://martin-helmer.com/Software/SegreClasses/index.html (also M2 Github)

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The algorithms resulting from the methods described here require general choices of constants. In practice this means a random choice of constants.

- Provided the constants are chosen from a sufficiently large set the probability of choosing a bad set of can be made arbitrarily small (this probability is analyzed for the special case of computing $s(X, \mathbb{P}^n)$ in a 2017 paper of Helmer & Schost).
- To make this deterministic, for a given pair X, Y we may determine a set of distinguished varieties which must be avoided. In Harris & Helmer 2018 we give an explicit description of these varieties (however the associated computation can be expensive).

In practical tests random choices from a set of constants of size 2^{15} has yielded correct answers on all over 10000 test runs.

Conclusion

We discussed methods to compute Segre classes $s(X, Y)$ for arbitrary $X \subset Y \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ (the ambient space could be any smooth projective toric variety). These also give algorithms to:

- Find the algebraic multiplicity of X inside Y.
- Test containment of X in the singular locus of Y.
- Test if an irreducible component of X is contained in some $Y \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$.
- Compute intersection products.

The main computational components of these algorithm can be implemented using any method to determine the number of solutions to a zero dimensional system of polynomial equations, i.e. we may use homotopy continuation (Bertini, PHCPack, ...), Gröbner bases (F4/F5), geometric resolutions (Kronecker), etc.