Segre-Driven Methods to find Algebraic Multiplicity & Test Ideal Membership

Martin Helmer
University of Copenhagen
joint work with Corey Harris [University of Oslo]
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Segre classes figure prominently in Fulton-MacPherson Intersection Theory, where they are seen to contain a wide variety of geometric and enumerative information regarding pairs of varieties (or schemes).

However using the information encoded in Segre classes to answer classical computational questions has previously been impractical.

We will discuss a new algorithm to compute Segre classes that leads to several practical computational methods which can:

• Test containment of varieties (i.e. prime ideal membership) without computing a Gröbner basis.
• Test if a variety $X$ is contained in the singular locus another variety $Y$ without finding the ideal of the singular locus of $Y$.
• Compute the algebraic multiplicity of subvarieties without working in local rings.
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- Test if a variety $X$ is contained in the singular locus another variety $Y$ without finding the ideal of the singular locus of $Y$.
- Compute the algebraic multiplicity of subvarieties without working in local rings.
The Segre class $s(X, Y)$ is a function of two varieties (or schemes) $X \subset Y$. We will focus on the case, $X \subset Y \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ (all results still hold if $X \subset Y \subset T_\Sigma$ for any smooth projective toric variety $T_\Sigma$).

Here Segre classes are polynomials in the Chow ring:

$$A^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}) \cong \mathbb{Z}[h_1, \ldots, h_m]/(h_1^{n_1+1}, \ldots, h_m^{n_m+1})$$

The $h_i$ represents the class of a general hyperplane in the $i^{th}$ factor $\mathbb{P}^{n_i}$. 

Example: Work in $\mathbb{P}^2 \times \mathbb{P}^3$ over $\mathbb{C}$ (or any algebraically closed field) and consider the hypersurface $Y = V(x_2^3x_0y_2^2 - x_3^3x_0y_2y_0 - x_3^3x_0y_2^0 - x_0^3x_0y_2^0)$. This hypersurface has multidegree $(3, 2)$ and its class is $[Y] = 3h_1 + 2h_2 \in A^*(\mathbb{P}^2 \times \mathbb{P}^3) \cong \mathbb{Z}[h_1, h_2]/(h_3^1, \ldots, h_4^2)$. Let $X = V(x_2^2x_0x_1y_2^2 - x_3^3x_0y_2y_0 - x_3^3x_0y_2^0 - x_0^3x_0y_2^0)$, then $X \subset Y$ and $s(X, Y) = -624h_2h_3 + 180h_2h_2 + 88h_1h_3 - 42h_2h_2 - 32h_1h_2 - 8h_2 + 6h_2 + 10h_1h_2 + 4h_2 + h_1$. 


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The $h_i$ represents the class of a general hyperplane in the $i^{th}$ factor $\mathbb{P}^{n_i}$.

**Example**

Work in $\mathbb{P}^2_x \times \mathbb{P}^3_y$ over $\mathbb{C}$ (or any algebraically closed field) and consider the hypersurface $Y = V(x_1^2x_0y_2^2 - x_0^3y_2y_0 - x_0^3y_0^2)$. This hypersurface has multidegree $(3, 2)$ and its class is

$$[Y] = 3h_1 + 2h_2 \in A^*(\mathbb{P}^2 \times \mathbb{P}^3) \cong \mathbb{Z}[h_1, h_2]/(h_1^3, h_2^4).$$

Let $X = V(x_0^2x_1y_1^2 - x_0^3y_0y_3, x_1^2x_0y_3^2 - x_0^3y_2y_0 - x_0^3y_0^2)$, then $X \subset Y$ and

$$s(X, Y) = -624h_1^2h_2^3 + 180h_1^2h_2^2 + 88h_1h_2^3 - 42h_1^2h_2 - 32h_1h_2^2 - 8h_2^3 + 6h_1^2 + 10h_1h_2 + 4h_2^2 + h_1$$
Segre classes can be used to find many interesting things:

- Chern-Schwartz-MacPherson ($c_{SM}$) class
- Euler characteristic
- Local Euler obstructions
- Samuel’s multiplicity of a subvariety
- Test if two ideals have the same integral closure
- Chern-Mather ($c_M$) class
- Euclidean distance degree
- Polar degrees
- Fulton & MacPherson’s intersection product in arbitrary projective varieties
- Containment of varieties

Segre classes have historically been difficult (or impossible) to compute. The first practical methods to compute Segre classes were due to Aluffi (2003) in the special case $s(X, \mathbb{P}^n)$, these were improved by Eklund, Jost, and Peterson (2011) and Helmer (2014). These methods were extended to the case $s(X, Y)$ for $X, Y$ subvarieties of $\mathbb{P}^n$ by Harris (2015). Our current work generalizes all previous methods, opening the way for new applications.
Let $X \subset Y \subset \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ be closed subschemes, choose generators of the ideal $I_X = (f_0, \ldots, f_r)$ defining $X$ so that all have the same multidegree.

The *projection of $Y$ from $X$* is the rational map:

$$pr_X : Y \dasharrow \mathbb{P}^r, \quad pr_X(p) = (f_0(p) : \cdots : f_r(p)).$$

The projective degrees of $X$ in $Y$, $g_a(X, Y)$, are

$$\sum_{i=0}^{\dim(Y)} \left[ pr_X^{-1}(\mathbb{P}^r-(\dim(Y)-i)) - X \right] = \sum_{|a|=a_1+\cdots+a_m \leq \dim(Y)} g_a(X, Y) \cdot h^{n-a} \in A^*(\mathbb{P}),$$

Projective degrees measure how algebraically dependent $f_0, \ldots, f_r$ are in $Y$. 

Proposition (Harris & Helmer, 2018)

The projective degrees of $X$ in $Y$ are given by

$$g_a(X, Y) = \deg(\mathbb{P}^r \cap L_a \cap W - X),$$

where $W = V(P_1, \ldots, P_{\dim(Y) - |a|})$, with $P_j = \sum \lambda_i f_i$ for general $\lambda_i \in \mathbb{C}$ and $L_a$ is a general linear space with class $h_a = h_a^1 \cdots h_a^m$. 
Let $X \subset Y \subset P = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ be closed subschemes, choose generators of the ideal $\mathcal{I}_X = (f_0, \ldots, f_r)$ defining $X$ so that all have the same multidegree.

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**Proposition (Harris & Helmer, 2018)**

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where $W = \nabla(P_1, \ldots, P_{\dim(Y)-|a|})$, with $P_j = \sum \lambda_i f_i$ for general $\lambda_i \in \mathbb{C}$ and $L^a$ is a general linear space with class $h^a = h_1^{a_1} \cdots h_m^{a_m}$. 
Let $X \subset Y \subset \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ be closed subschemes and let $(d_1, \ldots, d_m)$ be the (component wise) maximum multidegree of a set of generators for the ideal $\mathcal{I}_X$ defining $X$.

**Theorem (Harris & Helmer, 2018)**  
The Segre class $s(X, Y) \in A^*(\mathbb{P})$ is an explicitly given function of the projective degrees $g_a(X, Y)$ for $|a| = \dim(X), \ldots, 0$ and $(d_1, \ldots, d_m)$, the max multidegree of the generators of $\mathcal{I}_X$. 

**Corollary**  
The dimension $X$ part of the Segre class, $\{s(X, Y)\}_{\dim(X)}$, is completely determined by $(d_1, \ldots, d_m)$ and the projective degrees $g_a(X, Y)$ in dimension $X$, i.e. where $|a| = \dim(X)$.
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**Corollary**
The dimension $X$ part of the Segre class, $\{s(X, Y)\}_{\dim(X)}$, is completely determined by $(d_1, \ldots, d_m)$ and the projective degrees $g_a(X, Y)$ in dimension $X$, i.e. where $|a| = \dim(X)$. 
Let $R$ be the multigraded coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$, this ring has a $\mathbb{Z}^m$ grading with the first $n_1 + 1$ variables graded by $(1, 0, \ldots, 0)$, and so on.

Let $I$ be a prime ideal in $R$ defining a variety $X$ and let $I \supset J$ be a primary ideal in $R$, defining a scheme $Y$. The local ring of $Y$ along $X$ is defined as the localization of $R/J$ at the prime ideal $I$, that is

$$\mathcal{O}_{X,Y} = (R/J)_I.$$

Let $\mathcal{M}$ denote the maximal ideal of $\mathcal{O}_{X,Y}$. For $t >> 0$ and $d = \text{codim}(X, Y)$ the Hilbert-Samuel polynomial is

$$P_{HS}(t) = \ell(\mathcal{O}_{X,Y}/\mathcal{M}^t) = e_X Y \cdot \frac{t^d}{d!} + \text{lower terms}.$$

The coefficient $e_X Y$ of $\frac{t^d}{d!}$ is the algebraic multiplicity of $X$ on $Y$. 
The algebraic multiplicity $e_{X} Y$ is the integer coefficient of $[X]$ in $s(X, Y)$. 
Segre Classes and Samuel’s Algebraic Multiplicity

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**Example (Twisted Cubic V1)**

$R = \mathbb{C}[x, y, z, w]$ is the coordinate ring of $\mathbb{P}^3$, $A^*(\mathbb{P}^3) \cong \mathbb{Z}[h]/(h^4)$.

Define the prime ideal $I = (yw - z^2, xw - yz, xz - y^2)$

and the primary ideal $J = (z(yw - z^2) - w(xw - yz), xz - y^2)$, $J \subset I$.

Let $X = V(I)$, $Y$ the scheme associated to $J$ and note that $[X] = 3h^2$. We have

$$s(X, Y) = 6h^2 = 2[X] \in A^*(\mathbb{P}^3)$$

which gives that $e_X Y = 2$. 
Theorem (Harris & Helmer, 2018)
Let $Y \subset \mathbb{P}^n$ be a pure-dimensional subscheme and $X \subset Y$ a non-empty subvariety. Let $d$ be the maximum degree of the generators of $I_X$ and $I_Y$, and set $C = \dim(Y) - \dim(X)$. Then

$$e_X Y = \frac{\deg(Y)d^C - g}{\deg(X)} \in \mathbb{Z}_{>0},$$

where $g$ is the $\dim(X)$ projective degree of $X$ in $Y$.

A similar result holds in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. 
Theorem (Harris & Helmer, 2018)

Let $Y \subset \mathbb{P}^n$ be a pure-dimensional subscheme and $X \subset Y$ a non-empty subvariety. Let $d$ be the maximum degree of the generators of $\mathcal{I}_X$ and $\mathcal{I}_Y$, and set $C = \dim(Y) - \dim(X)$. Then

$$e_X Y = \frac{\deg(Y)d^C - g}{\deg(X)} \in \mathbb{Z}_{>0},$$

where $g$ is the $\dim(X)$ projective degree of $X$ in $Y$.

A similar result holds in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$.

Example (Twisted Cubic V2)

Again $X$ is the twisted cubic and $Y$ the ‘doubly twisted’ cubic. We have

$$e_X Y = \frac{\deg(Y)d^{\dim(Y) - \dim(X)} - g}{\deg(X)} = \frac{6 \cdot 3^0 - 0}{3} = 2.$$
Proposition (Samuel, 1955)

Let \( X \subseteq Y \) be subvarieties of \( \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m} \). Then \( e_X Y = 1 \) if and only if \( X \) is not contained in the singular locus of \( Y \).

Example (Containment of Varieties)

Work in \( \mathbb{P}^6 \), let \( X = \mathbb{V}(x_0, x_1, x_2, x_3, x_4) = \mathbb{V}(K) \) and let \( Y = \mathbb{V}(I) \) be an irreducible singular surface of degree 20 in \( \mathbb{P}^6 \) with

\[
I = 3 \times 3 \text{ minors of } \begin{pmatrix}
x_0 & x_1 & x_2 & x_3 \\
3x_3 & 4x_4 & 5x_5 & 6x_6 \\
x_2 & x_3 & x_4 & x_5 \\
x_0 + 5x_1 & x_1 + 6x_2 & x_2 + 7x_3 & x_3 + 8x_4
\end{pmatrix}.
\]

Is \( X \) contained in \( \text{Sing}(Y) \)?
Proposition (Samuel, 1955)

Let $X \subseteq Y$ be subvarieties of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. Then $e_X Y = 1$ if and only if $X$ is not contained in the singular locus of $Y$.

Example (Containment of Varieties)

Work in $\mathbb{P}^6_x$, let $X = \mathbb{V}(x_0, x_1, x_2, x_3, x_4) = \mathbb{V}(K)$ and let $Y = \mathbb{V}(I)$ be an irreducible singular surface of degree 20 in $\mathbb{P}^6$ with

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Is $X$ contained in $\text{Sing}(Y)$?

Yes, since $e_X Y = \frac{20 \cdot 3^{2-1} - g}{1} = \frac{60 - 58}{1} = 2 > 1$. 

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Is $X$ contained in $\text{Sing}(Y)$?
Yes, since $e_X Y = \frac{20 \cdot 3^{2-1} - g}{1} = \frac{60 - 58}{1} = 2 > 1$. This takes 0.07s.

Computing minors to get ideal $J$ defining $\text{Sing}(Y)$ takes 690s, reducing the generators of $J$ w.r.t. $K$ takes another 2s.
Let $X$ and $Y$ be arbitrary nonempty subschemes of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ defined by ideals $\mathcal{I}_X = (f_1, \ldots, f_r)$ and $\mathcal{I}_Y = (\omega_1, \ldots, \omega_s)$ (where all generators are chosen to have the same multidegree).

Let $\sum \lambda_i f_i$ and $\sum \gamma_j \omega_j$ be general $\mathbb{C}$-linear combinations defining hypersurfaces $\Theta_X$ and $\Theta_Y$, respectively.

**Theorem (Harris & Helmer, 2018)**

Let $Z = \Theta_X \cup \Theta_Y$. Then a top-dimensional irreducible component $V$ of $X$ is contained in $Y$ if and only if

$$\left\{ s(X, Z) \right\}_{\dim(X)} \neq \left\{ s(X, \Theta_X) \right\}_{\dim(X)}.$$
Segre Classes and Geometric Containment

Let $X$ and $Y$ be arbitrary nonempty subschemes of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ defined by ideals $I_X = (f_1, \ldots, f_r)$ and $I_Y = (\omega_1, \ldots, \omega_s)$ (where all generators are chosen to have the same multidegree).

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This solves ideal membership for prime ideals without a Gröbner basis.
Example (Containment in $\mathbb{P}_x^2 \times \mathbb{P}_y^2 \times \mathbb{P}_z^2$)

The irrelevant ideal is $B = (x_0, x_1, x_2) \cdot (y_0, y_1, y_2) \cdot (z_0, z_1, z_2)$. Let $F$ be a general polynomial of multidegree $(1, 1, 1)$ and let

$$f_1 = -10x_2y_1z_0 + 2x_1y_2z_0 + 35x_2y_0z_1 - 7x_0y_2z_1 - 25x_1y_0z_2 + 25x_0y_1z_2$$
$$f_2 = 9x_2y_1z_0 - 9x_1y_2z_0 - 4x_2y_0z_1 + 4x_0y_2z_1 + 3x_1y_0z_2 - 3x_0y_1z_2.$$

Consider $X = \mathbb{V}(\mathcal{I}_X)$, $Y = \mathbb{V}(\mathcal{I}_Y)$ where $\mathcal{I}_X = (y_0, y_1, y_2) \cdot (f_1, f_2) + (F)$, and $\mathcal{I}_Y = (z_0 \cdot f_1 - z_1 \cdot f_2, F)$.

Is the irreducible variety $X$ contained in $Y$?
Example (Containment in $\mathbb{P}_x^2 \times \mathbb{P}_y^2 \times \mathbb{P}_z^2$)

The irrelevant ideal is $B = (x_0, x_1, x_2) \cdot (y_0, y_1, y_2) \cdot (z_0, z_1, z_2)$. Let $F$ be a general polynomial of multidegree $(1, 1, 1)$ and let

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$$f_2 = 9x_2y_1z_0 - 9x_1y_2z_0 - 4x_2y_0z_1 + 4x_0y_2z_1 + 3x_1y_0z_2 - 3x_0y_1z_2.$$

Consider $X = \mathbb{V}(I_X)$, $Y = \mathbb{V}(I_Y)$ where $I_X = (y_0, y_1, y_2) \cdot (f_1, f_2) + (F)$, and $I_Y = (z_0 \cdot f_1 - z_1 \cdot f_2, F)$.

Is the irreducible variety $X$ contained in $Y$?

Yes, since $\{s(X, \Theta_X \cup \Theta_Y)\}_{\dim(X)} = 2[X] \neq [X] = \{s(X, \Theta_X)\}_{\dim(X)}$.

$\Theta_X$ is defined by a general linear combination of a multidegree $(1, 2, 2)$ set of gens for $I_X$, $\Theta_Y$ is defined by a general linear combination of multidegree $(1, 1, 2)$ gens for $I_Y$. 
Example (Containment in $\mathbb{P}^2_x \times \mathbb{P}^2_y \times \mathbb{P}^2_z$)

The irrelevant ideal is $B = (x_0, x_1, x_2) \cdot (y_0, y_1, y_2) \cdot (z_0, z_1, z_2)$. Let $F$ be a general polynomial of multidegree $(1, 1, 1)$ and let

$$f_1 = -10x_2y_1z_0 + 2x_1y_2z_0 + 35x_2y_0z_1 - 7x_0y_2z_1 - 25x_1y_0z_2 + 25x_0y_1z_2$$

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Consider $X = \mathbb{V}(I_X)$, $Y = \mathbb{V}(I_Y)$ where $I_X = (y_0, y_1, y_2) \cdot (f_1, f_2) + (F)$, and $I_Y = (z_0 \cdot f_1 - z_1 \cdot f_2, F)$.

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This takes 2.3s. Verifying $I_Y : B^\infty \subset I_X : B^\infty$ using Gröbner bases takes 11.4s.
Example (Equality of Irreducible Components (or radical ideals))

Work in $\mathbb{P}^6$, let $f_1 = x_2x_3x_5 - 5x_6^2x_0 + 3x_2x_0x_1$ and let $f_2$ be a general poly. of degree three. Consider the variety $X$ defined by $I_X = (f_1, f_2)$, and the irreducible scheme $Y$ defined by $I_Y = (f_1^2, f_1f_2, f_2^2)$.

We wish to verify that $I_X = \sqrt{I_Y}$. 
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$$Z = \mathbb{V}(\lambda_1 f_1 + \lambda_2 f_2) \cup \mathbb{V}(\lambda_3 f_1^2 + \lambda_4 f_1f_2 + \lambda_5 f_2^2) = \Theta_X \cup \Theta_Y$$ for general $\lambda_i \in \mathbb{C}$.

Computing the appropriate dim. $X$ (res. $Y$) projective degrees we obtain:

$$\{s(X, Z)\}_{\text{dim}(X)} = 9[X] \neq 3[X] = \{s(X, \Theta_X)\}_{\text{dim}(X)},$$

$$\{s(Y, Z)\}_{\text{dim}(X)} = 6[Y] \neq 4[Y] = \{s(Y, \Theta_Y)\}_{\text{dim}(X)}$$

Hence $X \subset Y$, $Y_{\text{red}} \subset X \Rightarrow X = Y_{\text{red}}$, and $I_X = \sqrt{I_Y}$ (takes 2.4s).

We used the fact that $X$ and $Y$ are irreducible. If we had only known $X$ was a variety $\Rightarrow$ an irreducible component of $Y$ was equal to $X$. 
Example (Equality of Irreducible Components (or radical ideals))

Work in $\mathbb{P}^6_x$, let $f_1 = x_2 x_3 x_5 - 5 x_6^2 x_0 + 3 x_2 x_0 x_1$ and let $f_2$ be a general poly. of degree three. Consider the variety $X$ defined by $I_X = (f_1, f_2)$, and the irreducible scheme $Y$ defined by $I_Y = (f_1^2, f_1 f_2, f_2^2)$.

We wish to verify that $I_X = \sqrt{I_Y}$.

$Z = \mathbb{V}(\lambda_1 f_1 + \lambda_2 f_2) \cup \mathbb{V}(\lambda_3 f_1^2 + \lambda_4 f_1 f_2 + \lambda_5 f_2^2) = \Theta_X \cup \Theta_Y$ for general $\lambda_i \in \mathbb{C}$.

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We used the fact that $X$ and $Y$ are irreducible. If we had only known $X$ was a variety $\Rightarrow$ an irreducible component of $Y$ was equal to $X$.

Note we use $I_X$ and $I_Y$ only; we do not compute $\sqrt{I_Y}$.

Computing $\sqrt{I_Y}$ to verify $I_X = \sqrt{I_Y}$ takes $> 8$ hours.
Goal: Understand the intersection of two projective varieties inside another, i.e. what does \( X \cap V \) look line inside \( Y \).

The intersection product \( X \cdot_Y V \) lets us do this.

**Example (Lines in a Quadric)**

Suppose that \( L_1 \) and \( L_2 \) are lines from the families of orange and purple lines (respectively) on the blue quadric \( Q \) in \( \mathbb{P}^3 \).

The intersection products \( L_1 \cdot_Q L_1 \) and \( L_1 \cdot_Q L_2 \) capture the behavior of the intersections of two orange lines and a orange and a purple line (respectively) *inside* the quadric \( Q \).
Let $Y \subset \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ be a nonsingular variety, let $X$ be regularly embedded in $Y$ and let $V$ by any subvariety of $Y$.

Fulton & MacPherson define the intersection product of $V$ by $X$ in $Y$ as

$$X \cdot_Y V = \{ c(N) \sim s(X \cap V, V) \}_{\text{expected dim.}} \in A^*(Y).$$

$N$ denotes the (pull back to $X$ of the) normal bundle $N_X Y$ (the vector bundle on $X$ with sections dual to $I_X/I_X^2$).

The product $X \cdot_Y V$ captures the behavior of the intersection $X \cap V$ inside of the variety $Y$.

$X \cdot_Y V$ is also the product of $[X]$ and $[V]$ in the Chow ring $A^*(Y)$.

It is often impossible to obtain an explicit form for $A^*(Y)$. Using Segre classes we can compute $X \cdot_Y V$ without knowing $A^*(Y)$. 
Example (Lines in a Quadric)

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Work in $A^*(\mathbb{P}^3) \cong \mathbb{Z}[h]/(h^4)$, $A^*(\mathbb{P}^3 \times \mathbb{P}^3) \cong \mathbb{Z}[h_1, h_2]/(h_1^4, h_2^4)$.

\[
L_1 \cdot Q L_1 = \{ c(T_Q) \backslash \Delta^*(s(L_1 \cap L_1, Q \times Q)) \}_{\dim 0}
= \{ c(T_Q) \backslash \Delta^*(h_1^3 h_2^2 + h_1^2 h_2^3 - 2h_1^3 h_2^3) \}^{(\nu)} = \left\{ \frac{(1 + h)^4}{1 + 2h} \cdot (h^2 - 2h^3) \right\}_{\dim 0} = 0 = \text{empty}
\]

\[
L_1 \cdot Q L_2 = \{ c(T_Q) \backslash \Delta^*(s(L_1 \cap L_2, Q \times Q)) \}_{\dim 0}
= \{ c(T_Q) \backslash \Delta^*(h_1^3 h_2^3) \}^{(\nu)} = \left\{ \frac{(1 + h)^4}{1 + 2h} \cdot h^3 \right\}_{\dim 0} = h^3 = \text{one point}
\]
Let $X \subset Y \subset \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ be closed subschemes and choose generators of the ideal $\mathcal{I}_X = (f_0, \ldots, f_r) \subset R = \mathbb{C}[x^{(1)}, \ldots, x^{(m)}]$ defining $X$ so that all have the same multidegree. Recall that the projective degrees of $X$ in $Y$ are $g_a(X, Y) = \deg (Y \cap L^a \cap W - X)$, where

$$W = \mathbb{V}(P_1, \ldots, P_{\dim(Y)-|a|}), \quad P_j = \sum \lambda_i f_i \text{ for general } \lambda_i \in \mathbb{C}.$$ 

One could compute this in several ways, in practice we use the method below.

**Theorem (Harris & Helmer, 2018)**

The projective degrees of $X$ in $Y$ are given by

$$g_a(X, Y) = \dim_\mathbb{C} \frac{R[T]}{\mathcal{I}_Y + \mathcal{I}_{L^a} + A + (P_1, \ldots, P_{\dim(Y)-|a|}, 1 - T \cdot P_0)},$$

where $A = (\ell(x^{(1)}) - 1, \ldots, \ell(x^{(m)}) - 1)$ for $\ell(x^{(i)})$ a general linear form in $x^{(i)}$ and $L^a$ is a general linear space with class $h^a$. 
The main computational steps of all our methods is to find the number of solutions to zero dimensional polynomial systems. The test implementation uses M2’s F4 GB; numeric methods could be used.

http://martin-helmer.com/Software/SegreClasses/index.html (also M2 Github)
Implementation

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The algorithms resulting from the methods described here require general choices of constants. In practice this means a random choice of constants.

- Provided the constants are chosen from a sufficiently large set the probability of choosing a bad set of can be made arbitrarily small (this probability is analyzed for the special case of computing $s(X, \mathbb{P}^n)$ in a 2017 paper of Helmer & Schost).
- To make this deterministic, for a given pair $X, Y$ we may determine a set of distinguished varieties which must be avoided. In Harris & Helmer 2018 we give an explicit description of these varieties (however the associated computation can be expensive).

In practical tests random choices from a set of constants of size $2^{15}$ has yielded correct answers on all over 10000 test runs.
Conclusion

We discussed methods to compute Segre classes $s(X, Y)$ for arbitrary $X \subset Y \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ (the ambient space could be any smooth projective toric variety). These also give algorithms to:

- Find the algebraic multiplicity of $X$ inside $Y$.
- Test containment of $X$ in the singular locus of $Y$.
- Test if an irreducible component of $X$ is contained in some $Y \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$.
- Compute intersection products.

The main computational components of these algorithm can be implemented using any method to determine the number of solutions to a zero dimensional system of polynomial equations, i.e. we may use homotopy continuation (Bertini, PHCPack, ...), Gröbner bases (F4/F5), geometric resolutions (Kronecker), etc.