Algorithms to Compute Chern-Schwartz-Macpherson and Segre Classes and the Euler Characteristic

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Overview

Let V be a closed subscheme of $\mathbb{P}^n = \mathrm{Proj}(k[x_0, \dots, x_n])$, with k an algebraical closed field of characteristic zero. Also let $A_*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1})$ denote the Chow ring of $\mathbb{P}^n$ where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ is the class of a hyperplane in $\mathbb{P}^n$.

- We give algorithms to compute the Chern-Schwartz-MacPherson class, Euler characteristic and Segre class of V.
- The algorithms can be implemented symbolically using Gröbner bases calculations, or numerically using homotopy continuation.
- The algorithms are tested on several examples and found to perform favourably compared to other existing algorithms.

Euler Characteristic

- The Euler characteristic χ provides a useful and interesting invariant for the study of a wide variety of topics, both in mathematics and in applications.
- The Euler characteristic of projective schemes has found recent applications to maximum likelihood estimation in algebraic statistics [5] and to string theory in physics [2].
- One may obtain the Euler characteristic of V, $\chi(V)$ directly from the Chern-Schwartz-MacPherson class of V $c_{SM}(V)$.
- In particular $\chi(V)$ is equal to the zero dimensional component of $c_{SM}(V)$ in $A_*(\mathbb{P}^n)$.

c_{SM} Classes

- In addition to containing the Euler characteristic c_{SM} classes are an important invariant in algebraic geometry, providing a generalization of the Chern class to singular schemes and also having desirable functorial properties and relationships to other common invariants.
- Note that by definition $c_{SM}(V) \in A_*(V)$, however we abuse notation and let $c_{SM}(V)$ denote the pushforward of $c_{SM}(V)$ to $\mathbb{P}^n$.

Projective Degree

Consider a rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$. We may define the *projective degrees* of the map ϕ as a list of integers $(g_0, \dots, g_n)$ where $g_i = \mathrm{card}\left(\phi^{-1}\left(\mathbb{P}^{m-i}\right) \cap \mathbb{P}^i\right)$. Here $\mathbb{P}^{m-i} \subset \mathbb{P}^m$ and $\mathbb{P}^i \subset \mathbb{P}^n$ are general hyperplanes of dimension $m-i$ and i respectively and card is the cardinality of a zero dimensional set.

Theorem 1

Let $I = (f_0, \dots, f_m)$ be a homogeneous ideal in $R = k[x_0, \dots, x_n]$ defining a ϱ-dimensional scheme $V = V(I)$, and assume, without loss of generality, that all the polynomials f_i generating I have the same degree. The projective degrees $(g_0, \dots, g_n)$ of $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, $\phi : p \mapsto (f_0(p) : \cdots : f_m(p))$, are given by

$$g_i = \dim_k\left(R[T]/(P_1 + \cdots + P_i + L_1 + \cdots + L_{n-i} + L_A + S)\right).$$

Here P_ℓ, L_ℓ, L_A and S are ideals in $R[T] = k[x_0, \dots, x_n, T]$ with $P_\ell = \left(\sum_{j=0}^m \lambda_{\ell,j} f_j\right)$ for $\lambda_{\ell,j}$ a general scalar in k, $S = \left(1 - T \cdot \sum_{j=0}^m \vartheta_j f_j\right)$, for ϑ_j a general scalar in k, L_ℓ a general homogeneous linear form for $\ell = 1, \dots, n$ and L_A a general affine linear form.

c_{SM} Class of a Hypersurface

Let $V = V(f) \subset \mathbb{P}^n$ be a hypersurface and let $(g_0, \dots, g_n)$ be the projective degrees of the polar map $\varphi : p \mapsto \left(\frac{\partial f}{\partial x_0}(p) : \cdots : \frac{\partial f}{\partial x_n}(p)\right)$.

Theorem 2.1 of Aluffi [1] gives the following in $A_*(\mathbb{P}^n)$

$$c_{SM}(V) = (1+h)^{n+1} - \sum_{j=0}^n g_j (-h)^j (1+h)^{n-j}. \quad (1)$$

Inclusion/Exclusion for c_{SM} Classes

For V_1, V_2 subschemes of $\mathbb{P}^n$ the inclusion-exclusion property for c_{SM} classes states $c_{SM}(V_1 \cap V_2) = c_{SM}(V_1) + c_{SM}(V_2) - c_{SM}(V_1 \cup V_2)$. Let $V = V(I) \subset \mathbb{P}^n$.

- Inclusion/Exclusion allows for the computation of $c_{SM}(V)$ for V of any codimension.
- This requires exponentially many c_{SM} computations relative to the number of generators of I.
- Must consider c_{SM} classes of products of many or all of the generators of I, which may have significantly higher degree than the original scheme V.

Segre Classes and Projective Degree

Consider a subscheme $Y = V(J) \subset \mathbb{P}^n$ defined by a homogeneous ideal $J = (w_0, \dots, w_m) \subset R = k[x_0, \dots, x_n]$. Assume, without loss of generality, that $\deg(w_i) = d$ for all i. Also let $(g_0, \dots, g_n)$ be the projective degrees of the rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, $\phi : p \mapsto (w_0(p) : \cdots : w_m(p))$.

By Proposition 3.1 of Aluffi [1] we have

$$s(Y, \mathbb{P}^n) = 1 - \sum_{i=0}^n \frac{g_i h^i}{(1+dh)^{i+1}} \in A_*(\mathbb{P}^n). \quad (2)$$

This together with Theorem 1 can be used to construct an algorithm to compute Segre classes.

Segre Class Algorithm

Our algorithm using (2) and Theorem 1 is denoted segre_proj_deg. Segre(Aluffi [1]) requires the computation of a certain Rees algebra, segreClass(E.J.P. [3]) requires the computation of the degrees of certain residual sets using a saturation. All algorithms are implemented in Macaulay2 [4].

Input	Segre(Aluffi [1])	segreClass(E.J.P. [3])	**segre_proj_deg(Theorem 1)**
Rational normal curve in $\mathbb{P}^7$	-	7s (9s)	8s (15s)
Smooth surface in $\mathbb{P}^8$ defined by minors	-	59s (-)	18s (-)
Degree 48 surface in $\mathbb{P}^6$	-	172s (-)	6s (-)
Singular var. in $\mathbb{P}^9$	0.5 s	33s (-)	10s (-)
Singular var. in $\mathbb{P}^6$	-	173s (-)	6s (-)
Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^3$ in $\mathbb{P}^{11}$	2s	- (-)	52s (-)

c_{SM} Class and Euler Characteristic Algorithm

Using (1) combined with Theorem 1 and inclusion/exclusion we obtain an algorithm to compute the c_{SM} class and Euler characteristic of an arbitrary subscheme of $\mathbb{P}^n$, denoted csm_polar.

All algorithms in the following table use inclusion/exclusion, all algorithms are implemented in Macaulay2 (M2); csm_polar is also implemented in Sage [8]. Euler is the built in M2 routine for computing Euler characteristics using Hodge numbers, it is only valid for non-singular schemes. CSM (Jost) is described in [6] and computes the degrees of certain residual sets using a saturation. CSM (Aluffi) is described in [1] and requires the computation of a certain Rees algebra.

Input	CSM (Aluffi)	CSM (Jost)	**csm_polar (M2)**	**csm_polar (Sage)**	euler
Twisted cubic	0.3s	0.1s (35s)	0.1s (37s)	0.1s (0.6s)	0.2s
Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$	0.4s	0.8s (148s)	0.2s (152s)	0.2s (57s)	0.2s
Smooth degree 3 surface in $\mathbb{P}^4$	-	1.2s (-)	0.6s (-)	0.2s (28s)	20.1s
Smooth degree 7 surface in $\mathbb{P}^4$	-	50s	42s	7.9s	-
Smooth degree 3 surface in $\mathbb{P}^8$	-	85.2s	2.2s	1.0s	-
Smooth degree 3 surface in $\mathbb{P}^{10}$	-	-	9.8s	2.0s	-
Sing. degree 3 surface in $\mathbb{P}^{10}$	-	-	10s	2.3s	n/a
Deg. 12 hypersurface in $\mathbb{P}^3$	25.3s	1.0s	0.2s	0.1s	n/a
Sing. Var. in $\mathbb{P}^5$	-	-	1.3s	0.3s	n/a

Notes on Example Computations

Test computations for the symbolic version were performed over $\mathbb{GF}(32749)$, yielding the same results found by working over $\mathbb{Q}$ for all examples considered. The symbolic methods are somewhat slower when performed over $\mathbb{Q}$, but still much faster than the numeric ones. Timings in () are for the numerical versions, using either Bertini [7] (via M2) or PHCpack [9] (via Sage). Computations that do not finish within 600s are denoted -.

References

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