Computing the Chern-Schwartz-MacPherson Class of a Complete Simplicial Toric Variety

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Overview

Consider complete simplicial toric variety X_Σ over the complex numbers $\mathbb{C}$ which is defined by a fan Σ.

- We give a combinatorial algorithm to compute the Chern-Schwartz-MacPherson class of X_Σ, $c_{SM}(X_\Sigma)$.
- This algorithm is based on a result of Barthel, Brasselet and Fieseler [3] which gives an expression for the c_{SM} class of a toric variety in terms of torus orbit closures.

Overview

Consider complete simplicial toric variety X_Σ over the complex numbers $\mathbb{C}$ which is defined by a fan Σ.

- Note that we restrict ourselves to complete simplicial toric varieties only for the purpose of simplifying the construction of the Chow ring of the toric variety.
- The statement of the result of Barthel, Brasselet and Fieseler [3] on which our algorithm is based does not require this restriction.
- If one was able to construct the Chow ring in a simple and meaningful manner with the restrictions removed the algorithm could be applied unchanged in this more general setting.

Chern-Schwartz-MacPherson Classes

- The c_{SM} class generalizes the Chern class of the tangent bundle to singular varieties/schemes, i.e. $c(T_V) \cdot [V] = c_{SM}(V)$ when V is a smooth subscheme of a smooth variety M.
- The c_{SM} class has important functorial properties, and relations to the Euler characteristic.
- More specifically in a 1974 article MacPherson [10] showed that the c_{SM} class is the unique natural transformation between the constructible function functor and the Chow group functor, thus proving a previous conjecture of Deligne and Grothendieck.
- The c_{SM} class is also closely related to the Euler characteristic.

Chern-Schwartz-MacPherson Classes

- That is the Euler characteristic is given by the degree of the zero dimensional component of $c_{SM}(V)$, i.e.

$$\int c_{SM}(V) = \chi(V).$$

- The Chern-Schwartz-MacPherson class has also been directly related to problems in string theory in Aluffi and Esole [2].

Chern-Schwartz-MacPherson Classes

- When V is a subscheme of $\mathbb{P}^n$ a result of Aluffi [**?**] tells us that the class $c_{SM}(V)$ can be thought of as a more refined version of the Euler characteristic since it contains the Euler characteristics of V and those of general linear sections of V for each codimension.
- Specifically, if $\dim(V) = m$, starting from $c_{SM}(V)$ we may directly obtain the list of invariants

$$\chi(V), \chi(V \cap L_1), \chi(V \cap L_1 \cap L_2), \ldots, \chi(V \cap L_1 \cap \cdots \cap L_m)$$

where $L_1, \ldots, L_m$ are general hyperplanes; this is an involution.

Previous/Current Algorithms to compute c_{SM} classes

- All previously written/implemented algorithms to compute c_{SM} classes have focused on subschemes of projective spaces $\mathbb{P}^n$ (except the algorithm of Helmer [5] which works for subschemes of some smooth toric varieties including products of projective spaces).
- The algorithm of Aluffi [1] works by computing blow-ups, the algorithm of Jost [8] works by computing saturations.
- The algorithms of Helmer [6] and [7] work by finding the vector space dimension of a polynomial ring mod a zero dimensional ideal.
- All these require solving problems involving polynomial systems.
- Hence a combinatorial algorithm gives a desirable speed boost where it is applicable.

The Chow Ring of a Complete Simplical Toric Variety

- For X_Σ a complete simplicial toric variety we will write $c_{SM}(X_\Sigma)$ as an element of the rational Chow ring $A^*(X_\Sigma)_\mathbb{Q}$.
- Similar to the case for smooth varieties we may construct the Chow ring from the Chow groups $A^j(X_\Sigma)$ of codimension j-cycles on X_Σ modulo rational equivalence.
- The only difference in this case will be that we work over the rational number field $\mathbb{Q}$ rather than the integers.
- The rational Chow ring of X_Σ is given by the graded ring

$$A^*(X_\Sigma)_\mathbb{Q} = A^*(X_\Sigma) \otimes_\mathbb{Z} \mathbb{Q} = \bigoplus_{j=0}^{n} A^j(X_\Sigma) \otimes_\mathbb{Z} \mathbb{Q}.$$

The Chow Ring of a Complete Simplical Toric Variety

- For a cone $\sigma \in \Sigma$ we will write

$$[V(\sigma)] \in A^{\dim(\sigma)}(X_\Sigma)$$

for the rational equivalence class of the orbit closure $V(\sigma) = \overline{O(\sigma)}$.

- The collections

$$[V(\sigma)] \in A_j(X_\Sigma)$$

for $\sigma \in \Sigma$ having dimension $n - j$ generate $A_j(X_\Sigma)$, the Chow group of dimension j.

- Further the collection $[V(\sigma)]$ for all $\sigma \in \Sigma$ generates $A^*(X_\Sigma)$ as an abelian group.

The Chow Ring of a Complete Simplical Toric Variety

Let X_Σ be a complete and simplicial toric variety with generating rays $\Sigma(1) = \rho_1, \ldots, \rho_r$ where $\rho_j = \langle v_j \rangle$ for $v_j \in N$. Then we have that:

$$\mathbb{Q}[x_1, \ldots, x_r]/(\mathcal{I} + \mathcal{J}) \cong A^*(X_\Sigma)_\mathbb{Q},$$

with the isomorphism map specified by $[x_i] \mapsto [V(\rho_i)]$.

- Here $\mathcal{I}$ denotes the Stanley-Reisner ideal of the fan Σ,

$$\mathcal{I} = (x_{i_1} \cdots x_{i_s} \mid i_{i_j} \text{ distinct and } \rho_{i_1} + \cdots + \rho_{i_s} \text{ is not a cone of } \Sigma)$$

- $\mathcal{J}$ denotes the ideal of $\mathbb{Q}[x_1, \ldots, x_r]$ generated by the linear relations of the rays.

The c_{SM} Class in Terms of Orbit Closures

Proposition 1 (Main Theorem Barthel, Brasselet and Fieseler [3])

Let X_Σ be an n-dimensional complex toric variety specified by a fan Σ. We have that the Chern-Schwartz-MacPherson class of X_Σ can be written in terms of orbit closures as

$$c_{SM}(X_\Sigma) = \sum_{\sigma \in \Sigma} [V(\sigma)] \quad \in A^*(X_\Sigma)_{\mathbb{Q}} \tag{1}$$

where $V(\sigma)$ is the closure of the torus orbit corresponding to σ.

- This result will allows us to compute c_{SM} classes using a computer algebra system.

Computing the Multiplicity of a Cone

Lemma 2 (Modified version of 11.1.8. of Cox, Little, Schenck [4])

Let $N = \mathbb{Z}^n$ *be an integer lattice. For a simplicial cone* $\sigma = \rho_1 + \cdots + \rho_d \subset N$ *let* $\mathfrak{M}_\sigma$ *be the matrix with columns specified by the generating vectors of the rays* $\rho_1, \ldots, \rho_d$ *which define the cone* σ*; we have*

$$\mathrm{mult}(\sigma) = |\mathsf{det}(\mathrm{Herm}(\mathfrak{M}_\sigma))| \tag{2}$$

where $\mathrm{Herm}(\mathfrak{M}_\sigma)$ *denotes the Hermite normal form of matrix* $\mathfrak{M}_\sigma$ *with all zero rows and/or zero columns removed.*
Further $\mathrm{mult}(\sigma) = 1$ *if and only if* σ *is a smooth cone.*

Computing the Orbit Closures

Proposition 3 (Theorem 12.5.2. of Cox, Little, Schenck [4])

Assume that X_Σ *is complete and simplicial. If* $\rho_1, \dots, \rho_d \in \Sigma(1)$ *are distinct and if* $\sigma = \rho_1 + \cdots + \rho_d \in \Sigma$ *then in* $A^*(X_\Sigma)$ *we have the following:*

$$[V(\sigma)] = \mathrm{mult}(\sigma)[V(\rho_1)] \cdot [V(\rho_2)] \cdots [V(\rho_d)]. \qquad (3)$$

Here $\mathrm{mult}(\sigma)$ *will be calculated using Lemma 2.*

- This gives us a method to compute the classes of the orbit closures $[V(\sigma)]$ for $\sigma \in \Sigma$.
- We are now ready to construct an algorithm to compute the c_{SM} class and Euler characteristic of a complete simplicial toric variety.

The c_{SM} class of a Complete Simplicial Toric Variety

Input: A complete, simplicial toric variety X_Σ defined by a fan Σ with $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$. We assume $\dim(X_\Sigma) \geq 1$.
Output: $c_{SM}(X_\Sigma)$ in $A^*(X_\Sigma)_{\mathbb{Q}} \cong \mathbb{Q}[x_1, \ldots, x_r]/(\mathcal{I} + \mathcal{J})$.

- Compute rational Chow ring $A^*(X_\Sigma)_{\mathbb{Q}} \cong \mathbb{Q}[x_1, \ldots, x_r]/(\mathcal{I} + \mathcal{J})$.
- $\mathrm{csm} = 0$.
- **For** i **from** $\dim(X_\Sigma)$ **to** 1**:**
  - $\mathrm{orbits} =$ all subsets of $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$ containing i elements.
  - $\mathrm{total} = 0$.
  - **For** $\rho_{j_1}, \ldots, \rho_{j_i}$ **in** orbits**:**
    - $[V(\rho_{j_1} + \cdots + \rho_{j_i})] = [V(\sigma)] = \begin{cases} 0 \text{ if } \sigma \notin \Sigma \\ \mathrm{mult}(\sigma) x_{j_1} \cdots x_{j_i} \text{ otherwise} \end{cases}$
    - $\mathrm{total} = \mathrm{total} + [V(\sigma)]$.
  - $\mathrm{csm} = \mathrm{csm} + \mathrm{total}$.
- **Return** $c_{SM}(X_\Sigma) = csm$.

The c_{SM} class of a Complete Simplicial Toric Variety
Example of Our Algorithm

- Compute $\chi(\mathbb{P}^3)$ and $c_{SM}(\mathbb{P}^3) = c(T_{\mathbb{P}^3}) \cdot [\mathbb{P}^3]$ in $A^*(\mathbb{P}^3)$.
- As a toric variety $\mathbb{P}^3 = X_\Sigma$ is specified by a fan Σ with rays $\rho_0 = \langle(1,0,0)\rangle$, $\rho_1 = \langle(0,1,0)\rangle$, $\rho_2 = \langle(0,0,1)\rangle$, $\rho_3 = \langle(-1,-1,-1)\rangle$ (and maximal cones given by the cones of all 3 element subsets of the rays above).
    - The codimension one part of $c_{SM}(\mathbb{P}^3)$ is

$$
\begin{aligned}
(c_{SM}(\mathbb{P}^3))^{(1)} &= \mathrm{mult}(\rho_0)[V(\rho_0)] + \mathrm{mult}(\rho_1)[V(\rho_1)] \\
&+ \mathrm{mult}(\rho_2)[V(\rho_2)] + \mathrm{mult}(\rho_3)[V(\rho_3)] \in A^*(\mathbb{P}^3) \\
&= x_0 + x_1 + x_2 + x_3 \\
&= 4x_3 \in A^*(\mathbb{P}^3) \cong \mathbb{Z}[x_3]/(x_3^4).
\end{aligned}
$$

- Similarly we compute $c_{SM}(\mathbb{P}^3))^{(2)}$ and $c_{SM}(\mathbb{P}^3))^{(2)}$ giving $c_{SM}(\mathbb{P}^3) = 4x_3^3 + 6x_3^2 + 4x_3 + 1$.

The c_{SM} class of a Complete Simplicial Toric Variety

- The † denotes that for these versions of the algorithms the input is checked for smoothness.
- If the input is found to be smooth we know $\mathrm{mult}(\sigma) = 1$ for all cones $\sigma \in \Sigma$ and hence we do not compute Hermite normal forms and determinates.

Input	c_{SM} Alg. †	c_{SM} Alg.	Chow Ring
$\mathbb{P}^6$	0.0s	0.0s	0.1 s
$\mathbb{P}^{12}$	0.2s	3.8s	0.3 s
$\mathbb{P}^{16}$	5.3s	85.4s	0.7 s
$\mathbb{P}^5 \times \mathbb{P}^6$	0.3s	3.7s	1.2 s
$\mathbb{P}^5 \times \mathbb{P}^8$	1.1s	16.8s	2.1 s
$\mathbb{P}^8 \times \mathbb{P}^8$	12.0s	168.5s	4.5 s
$\mathbb{P}^5 \times \mathbb{P}^5 \times \mathbb{P}^5$	12.8s	156.7s	11.8 s
$\mathbb{P}^5 \times \mathbb{P}^5 \times \mathbb{P}^6$	28.4s	387.1s	17.0 s
Fano sixfold 123	0.3s	1.0s	1.1 s
Fano sixfold 1007	0.4s	1.0s	1.8 s

Discussion

- By default the M2 implementation checks if the input defines a smooth toric variety, i.e. performs the procedure of the implementations marked with †.
- The comparison between the two implementations shows how the algorithm would perform on a singular input of a similar size and complexity (since the implementation without the † always computes the Hermite forms and their determinates in Lemma 2).
- In this way we see in a precise manner what the extra cost associated to computing the c_{SM} class of a singular toric variety would be in comparison to the cost of computing a smooth toric variety defined by a fan having similar combinatorial structure.

Discussion

- Note that the extra cost in the singular case comes entirely from performing linear algebra with integer matrices.
- As such the running times in these cases may be somewhat reduced by using a specialized integer linear algebra package.
- To give a rough quantification of what performance improvement one might expect we performed some testing using LinBox [9] and PARI [12] via Sage [11] on linear systems of similar size and structure to those arising in the examples in table.
- In this testing we found that the specialized algorithms seemed to be around two to three times faster than the linear algebra methods used by our implementation in the “CharToric” package, however this testing is by no means conclusive.

Demo: An M2 Package to compute Characteristic Classes

- For toric varieties we use the "NormalToricVarieites" package to define the variety.
- Our package can also compute c_{SM} and Segre classes of subschemes of some smooth toric varieties.
- These features will be available in the "CharacteristicClasses" package in the next release of M2.

Future/Current Work

- Investigate if the c_{SM} class of a toric variety or a subscheme of a toric variety can be related to the Euler characteristics of general linear sections in an analogous way to the $\mathbb{P}^n$ case.
- Make improvements to the M2 implementation of the algorithm.
- Investigate if it is possible to generalize the algorithms to other toric varieties in an efficient and meaningful way.
- Investigate applications of the algorithm.
- Subschemes of more general toric varieties (working with subschemes requires solving polynomial systems however).

Thank you for listening!

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